QUALIFYING EXAMINATION

January 2010

MA 554

1. (15 points) Let R be a ring (commutative, with identity), M an R-module, and N a submodule of M. Write $\iota : N \hookrightarrow M$ for the natural inclusion map and $-^* = \mathrm{Hom}_R(-, R)$ for R-duals.

(a) Show that if M/N is free, then $\iota^* : M^* \to N^*$ is surjective.

(b) Give an example showing that the assumption of freeness is needed in part (a).

2. (17 points) Let M be a finitely generated $\mathbb{Z}$-module (i.e., a finitely generated Abelian group).

(a) Let r denote the rank of M (i.e., $M \simeq \mathbb{Z}^r \oplus T$ with T a torsion module). Show that r is the maximal number of linearly independent elements in M.

(b) Let N be a (necessarily finitely generated) submodule of M. Show that if M/N is a torsion module, then M and N have the same rank.

3. (16 points) Consider the matrix

$$A = \begin{pmatrix} 0 & 1 & 0 & x^3+x & x+1 \\ -x & x^2-x & x & 0 & 0 \\ 0 & -1 & 0 & -x^3 & -x-1 \\ x^3-x^2+x & x^3-x^2 & x^2 & 0 & 0 \end{pmatrix}$$

with entries in the polynomial ring $R = \mathbb{Q}[x]$. Determine the dimension of the cokernel of A, considered as a vector space over $\mathbb{Q}$. (Recall that A defines an R-linear map $R^5 \longrightarrow R^4$ and that every R-module is a vector space over $\mathbb{Q}$ via the inclusion $\mathbb{Q} \subset R$.)

4. (11 points) Determine all positive integers n such that there exists an n by n matrix A with coefficients in $\mathbb{Q}$ satisfying $A^3 = 2 \cdot I_n$. (Here I_n denotes the n by n identity matrix.)

5. (14 points) Consider the elementary Jordan matrix

$$A = \begin{pmatrix} 0 & & & & \\ 1 & \cdot & & & \\ & \cdot & \cdot & & \\ & & \cdot & \cdot & \\ & & & 1 & 0 \end{pmatrix}$$

of size n by n over a field K. Determine the Jordan canonical form of A^2.

6. (12 points) Let R be a domain and A an n by n matrix with entries in R, where $n \geq 2$. Prove that $\det(\mathrm{adj}(A)) = (\det(A))^{n-1}$. (Recall that $\mathrm{adj}(A)$ is the n by n matrix whose (i, j)-entry is $(-1)^{i+j}$ times the determinant of the matrix obtained from A by deleting row j and column i.)

7. (15 points) Let V be a finite-dimensional vector space over $\mathbb{C}$. Show that f is a symmetric bilinear form of rank at most 2 on V if and only if there exist φ and ψ in V^* such that $f(x, y) = \varphi(x)\psi(y) + \varphi(y)\psi(x)$ for every x and y in V.