**QUALIFYING EXAMINATION**
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MATH 554 - Prof. J. Wang

1. Let $E_i$, $i = 1, \cdots, \ell$ be projections of a finite dimensional vector space $V$ and $\alpha_1, \cdots, \alpha_\ell$ scalars. If $E_i$ have the same range, then $A = \sum_{i=1}^{\ell} \alpha_i E_i$ satisfies $A^2 = \left( \sum_{i=1}^{\ell} \alpha_i \right) A$.

2. Let $F$ be a field and $S = \{E \in M_{nn}(F) \mid E^2 = E\}$. Show that the span of $S$ is $M_{nn}(F)$.

3. Let $A = [A_1, A_2, A_3, A_4, A_5]$ be a $4 \times 5$ matrix. Assume that the general solution for $AX = 0$ is given by

$$X = \begin{bmatrix} s \\ 2s - t \\ t \\ t + s \\ 2t \end{bmatrix}.$$

   (a) Find a maximal independent subset $B$ of $\{A_1, A_2, A_3, A_4, A_5)$.

   (b) Express $A_1 + A_2 + A_3 + A_4 + A_5$ as a linear combination of $B$.

4. Let $K$ be a field and $D : M_{nn}(K) \to K$ be a function such that $D(AB) = D(A)D(B)$ and $D(I) \neq D(0)$. Show that if $\text{rank}(A) < n$, then $D(A) = 0$.

5. Let $R$ be a commutative ring with identity, $A \in M_{mn}(R)$, $B \in M_{nm}(R)$ and $I$ the identity matrix.

   (a) $|I_m - AB| = |I_n - BA|$.

   (b) If $R$ is a field and $n \leq m$, show that the characteristic polynomials $p_{AB}(x)$ and $p_{BA}(x)$ of $AB$ and $BA$ respectively satisfy $p_{AB}(x) = x^{m-n} p_{BA}(x)$.

6. Let $A = [a_{ij}]$ be the $(n+1) \times (n+1)$ matrix with $a_{ij} = (i + j - 2)!$ and $0! = 1$. (Hint: $A = LDL^T$)

   (a) $A$ is positive definite.

   (b) $\det A = (0!1! \cdots n!)^2$.

   (c) $(n!)^2 A^{-1} \in M_{(n+1)(n+1)}(\mathbb{Z})$.

7. Let $A$ be an $n \times n$ matrix over $\mathbb{C}$ and $p$ be a prime number. Suppose that $I \neq A$ and $A^p = I$ and $\mathrm{tr}(A) =$ positive integer $\ell$. Show that $n = \ell + sp$ with $s$ a positive integer.