Math 554 Qualifying Examination

January 2004—Prof. Lipman

- *Each question 1–6 is worth 10 points (60 total).*
- *In answering any part of a question you may assume the preceding parts.*
- *Please begin your answer to each of the six questions on a new sheet of paper.*

1. Let V_1 and V_2 be subspaces of a fixed vector space V (not necessarily finite-dimensional) over a field. For any subspace $W \subset V$ denote by $\bar{v}$ the image of $v \in V$ under the canonical linear map $V \to V/W$.

(a) Prove carefully the existence of a linear map $p\colon (V/V_1) \oplus (V/V_2) \to V/(V_1 + V_2)$ such that $p(\bar{x}, \bar{y}) = \overline{x - y}$.

"Carefully" means that your answer should be completely understandable to students whose knowledge about vector spaces includes nothing more than direct sums plus whatever is needed to follow the above definition of $\bar{v}$.

(b) Prove that there is an isomorphism from $V/(V_1 \cap V_2)$ to the kernel of p.

(c) Suppose V_1, V_2 have finite codimension in V. With "codim" denoting codimension, what simple relation among $\operatorname{codim}(V_1 \cap V_2)$, $\operatorname{codim}(V_1)$, $\operatorname{codim}(V_2)$ and $\operatorname{codim}(V_1 + V_2)$ results from (b)?

2. For any $n \times n$ complex matrix A set

$$e^A := \sum_{k=0}^{\infty} \frac{1}{k!} A^k.$$

(It is not hard to show, and you may assume, that the series converges when you think of $n \times n$ matrices as being points in $\mathbb{C}^{n^2}$.) Prove the following statements:

(a) For any invertible complex $n \times n$ matrix B,

$$e^{B^{-1}AB} = B^{-1} e^A B.$$

<u>Hint</u>. It is easily seen, and you may assume, that for fixed B, $B^{-1}CB$ is a continuous function of C.

(b)
$$\det(e^A) = e^{\operatorname{trace}(A)}.$$

3. Find the characteristic polynomial, the minimal polynomial, the rational canonical form, and the Jordan canonical form of the matrix

$$\begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

4. (a) Let V be a finite-dimensional vector space over a field k, and let $L\colon V \to V$ be a linear map. Prove that if $F(X) \in k[X]$ is a polynomial relatively prime to the characteristic polynomial of L then the linear map $F(L)$ is invertible.

Suppose further that $k = \mathbb{C}$, that V is an inner product space, and that L is self-adjoint. Let I be the identity map of V. Prove:

(b) $I + iL$ is invertible $\quad (i := \sqrt{-1})$.

(c) $U := (I - iL)(I + iL)^{-1}$ is unitary.

5. Prove or disprove each of the following statements:

(a) Let A and B be $n \times n$ complex matrices which have the same characteristic and minimal polynomials, and such that for each root λ of the characteristic polynomial, the nullspaces of $(A - \lambda I)$ and $(B - \lambda I)$ $(I :=$ identity matrix$)$ have the same dimension. Then A and B are similar.

(b) Let A and B be *normal* $n \times n$ complex matrices which have the same characteristic polynomial. Then there exists a *unitary* matrix U such that $UAU^{-1} = B$.

6. For a finitely-generated torsion module M over a principal ideal domain R, with invariant factors $a_1|a_2|\cdots|a_s$, let $\chi(M)$ be the ideal $a_1 a_2 \ldots a_s R$. (Set $\chi\big((0)\big) := R$.)

Prove that if $\chi(M) \subset bR$ $(b \in R)$ then $bR = \chi(N)$ for some R-submodule $N \subset M$.