MATH 554 QUALIFYING EXAM

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• Please begin each question 1–5 on a new sheet.

• In doing any question, you may assume preceding parts of that question, even if you haven't done them.

• You may quote without proof any standard fact included in the MA554 syllabus.

1. [10 pts] Let M be the matrix $\begin{bmatrix}1&2&1\\0&1&0\\0&0&1\end{bmatrix}$ (with entries in the rational field $\mathbf{Q}$).

Find an invertible matrix U such that $U^{-1}MU$ has rational canonical form.

Hint. First find the canonical matrix C, then solve $MU = UC$.

2. [10 pts] A matrix A over $\mathbf{C}$ has characteristic polynomial $X^2(X-4)^6$ and minimal polynomial $X(X-4)^3$. Find *all* possibilities for the geometric multiplicities of each of the eigenvalues of A.

The *geometric multiplicity* of the eigenvalue λ is the dimension of the nullspace of $A - \lambda I$.

3. [10 pts] *Notation*: Let k be a field, and T an indeterminate. $M_n(k)$ is the ring of $n \times n$ matrices with entries in k; $I_n \in M_n(k)$ is the identity matrix; and $k(T)$ is the field of all quotients of polynomials $f(T)/g(T)$ with $g(T) \neq 0$ (where $f(T)/g(T) = f'(T)/g'(T) \iff$ the polynomials fg' and $f'g$ are equal).

(a) Show: if $B \in M_n(k)$ then $B - TI_n \in M_n\big(k(T)\big)$ is invertible.

(b) Show: if $A, B \in M_n(k)$ then AB and BA have the same characteristic polynomial.

Hint. Do it first with $B - TI_n$ in place of B.

4. [25 pts] Let E be a finite-dimensional inner product space over $\mathbf{C}$, $F \subset E$ a subspace, and p orthogonal projection onto F (considered as a linear map from E into itself).

(a) Prove that a linear map $T\colon E \to E$ commutes with $p \iff F$ is invariant under both T and its adjoint T^* (i.e., $T(F) \subset F$ and $T^*(F) \subset F$).

(b) Prove: If T is normal, then T commutes with $p \iff F$ is T-invariant.

(c) Prove: If T is normal, with spectral decomposition $T = \sum \lambda_i p_i$ ($\lambda_i \in \mathbf{C}$), then T commutes with p if and only if p_i commutes with p for all i.

5. [25 pts] (a) Let V be an inner product space of finite dimension n over $\mathbf{C}$, and let W be an m-dimensional subspace of V. Prove that there exists an orthonormal basis $(v_1, v_2, \dots, v_n)$ of V such that $(v_1, v_2, \dots, v_m)$ is a basis of W.

(b) Let M be an $n \times r$ complex matrix of rank m. Prove that there exists a unitary $n \times n$ matrix Q and an upper triangular $n \times r$ matrix R whose bottom $n - m$ rows consist entirely of zeros, such that $M = QR$.

Definition: $R = (r_{ij})$ is *upper triangular* if $r_{ij} = 0$ whenever $i > j$.
Hint: There is an orthonormal basis $\mathbf{v} = (v_1, v_2, \dots, v_n)$ of $\mathbf{C}^n$ whose first m members come from applying the Gram-Schmidt process to the columns of M. (Why?) Let Q be the matrix whose columns are the v_i ...

(c) For $M = \begin{bmatrix}1&2&-1\\0&1&0\\1&0&-1\end{bmatrix}$ find Q and R as in (b).