Qualifying Examination
August, 2000
Math 554 - Prof. Moh

You have to show your work.

Notation: Let $\mathbb{Z}$ be the ring of integers, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$ be the fields of rational, real, complex numbers respectively. Let i be the pure imaginary number $\sqrt{-1}$. Let $\mathbb{K}$ be a field and $\mathbb{M}_n(\mathbb{K})$ be the set of $n \times n$ matrices with elements from $\mathbb{K}$. Let $\mathbb{V}$ be an n-dimensional vector space over $\mathbb{K}$. Let α be a linear operator on V.

1. Let w be the complex number $e^{2\pi i/8}$, and the 8 by 8 complex matrix A be defined as $A = (a_{ij})$ with $a_{ij} = w^{(i-1)(j-1)}$.

(a): Is A diagonalizable? [10 points]

(b): Find all eigenvalues of A. [10 points]

2. A rotation of $\mathbb{R}^n$ is defined to be a linear transformation of $\mathbb{R}^n$ which preserves the length of all vectors and the orientation (i.e., with positive determinant). Let A be a rotation of $\mathbb{R}^3$, show that 1 is an eigenvalue of A. [10 points]

3. Express the commutative group $\mathbb{Z}^3/(f_1, f_2, f_3)$ where $f_1 = (2,4,6)$, $f_2 = (4,6,8)$, $f_3 = (3,4,5)$ as a direct sum of cyclic groups. [10 points]

4. Let p be a prime number, and $A = \mathbb{Z}/(p^2) \oplus \mathbb{Z}/(p^2) \oplus \mathbb{Z}/(p^3)$. Compute the number of non-cyclic subgroups of A of order p^2. [10 points]

5. Let $A \in \mathbb{M}_n(\mathbb{K})$. If A is nilpotent, then $A^n = 0$. [10 points]

6. Find the area of the convex pentagon in $\mathbb{R}^2$ with vertices $(0,0),(5,0),(7,3),(4,6),(0,4)$. [10 points]

7. Let $\mathbb{P}_3$ be the vector space of all real polynomials of degree 3 or less. Let the inner product $(f|g)$ be defined as $\int_0^1 fg\,dx$. Find an orthonormal basis of $\mathbb{P}_3$. [10 points]

8. Find the Jordan canonical form of the following matrix over complex numbers

$$\begin{pmatrix} -1 & 1 & 1 \\ -3 & 2 & 2 \\ -1 & 1 & 1 \end{pmatrix}$$

[10 points]

9. Let A be an $n \times n$ normal complex matrix. Show that A^* is a polynomial in A.
[10 points]