QUALIFYING EXAMINATION
JANUARY, 1997
MATH 554

In answering any part of a question you may assume the preceding parts.

NOTATION: V is a finite dimensional vector space over a field K;
$\alpha\colon V \to V$ is a linear operator.

1. In some basis of V, α is given by the matrix $A = \begin{bmatrix} 1 & 1 & -2 & 0 \\ 2 & 1 & 0 & 2 \\ 1 & 0 & 1 & 1 \\ 0 & -1 & 2 & 1 \end{bmatrix}$. Find:

(1) the rational normal form of α. [8]

(2) the Jordan normal form of α. [7]

2. Let P be the space of polynomials of degree $< n$ over K, and let $\delta\colon P \to P$ be the operator, given by differentiation: $\delta\big(\sum_{i=0}^{n-1} a_i x^i\big) = \sum_{i=1}^{n-1} i a_i x^{i-1}$.

Find the Jordan normal form of the δ^2, when

(1) K is the field $\mathbb{C}$ of complex numbers. [5]

(2) K is the field $\mathbb{F}_3$ with 3 elements. [5]

3. A α-invariant subspace $W \leq V$ is called irreducible, if the only proper α-invariant subspaces of W are 0 and W itself.

(1) Prove that if the characteristic polynomial of α has an irreducible factor of degree d, then α has an irreducible invariant subspace of dimension d. [10]

(2) Prove the converse of (1). [10]

4. Let v_1, v_2 and w_1, w_2 be two pairs of vectors in a real inner product space V.

(1) Prove that if $\|v_1\| = \|w_1\|$, $\|v_2\| = \|w_2\|$, and $\angle(v_1, v_2) = \angle(w_1, w_2)$, then there is an orthogonal operator $\alpha\colon V \to V$, such that $\alpha(v_1) = w_1$ and $\alpha(v_2) = w_2$. [10]

(2) Give necessary and sufficient conditions for α in (1) to be unique. [5]

(3) Does the converse of (1) hold? [5]

5. Let B be the subgroup of $\mathbb{Z}^3$ generated by $(3, 6, 3)$, $(-1, 4, 0)$, and $(5, 4, 6)$, and let $A = \mathbb{Z}^3/B$.

(1) Express A as a direct sum of cyclic groups. [8]

(2) How many distinct subgroups of order 6 does A contain? [7]

6. Let A be a finite abelian group, let n be an integer, and let $\beta\colon A \to A$ be the map, defined by $\beta(a) = na$ for each $a \in A$.

(1) Prove that the abelian groups $\operatorname{Ker}(\beta)$ and $A/\operatorname{Im}(\beta)$ are isomorphic. [10]

(2) Prove that for each prime number p, the number of subgroups of A of order p is equal to the number of subgroups of A of index p.

[*Hint*: Use the preceding problem with $n = p$.] [10]