Math 554 Qualifying Examination
August, 1997

In answering any part of a question you may assume the preceding parts.

1. Let A be an 8×8 complex matrix satisfying the following conditions (with I an 8×8 identity matrix):

(i) $\text{Rank}(A+I) = 6$, $\text{rank}(A+I)^2 = 5$, and $\text{rank}(A+I)^k = 4$ for all $k \geq 3$.
(ii) $\text{Rank}(A-2I) = 7$ and $\text{rank}(A-2I)^k = 6$ for all $k \geq 2$.
(iii) $\text{Rank}(A-3I)^k = 6$ for all $k \geq 1$.

Find the Jordan form of A.

2. (a) Prove that any linear transformation T of a finite-dimensional $\mathbb{C}$-vector space V can be written in the form $T = S + N$ where S is diagonalizable, N is nilpotent, and $SN = NS$. (Hint: Look at the Jordan form.)

(b) Use (without proof) the equivalence of "S diagonalizable" and "the minimal polynomial of S has no multiple factors" to show that if S is diagonalizable and if W is an S-invariant subspace of V then the restriction of S to W is diagonalizable.

(c) Let $T = S + N$ be as in (a). Let I be the identity transformation of V. For any $\lambda \in \mathbb{C}$, and any $m > 0$, let $W_{\lambda,m}$ be the kernel of $(T - \lambda I)^m$. Show that the restriction of $(S - \lambda I)$ to $W_{\lambda,m}$ is a diagonalizable, nilpotent, linear transformation of $W_{\lambda,m}$ into itself, and that such a transformation must be zero.

(d) Using (c), or otherwise, show that S and N in (a) are uniquely determined by T.

3. Let E be an n-dimensional complex vector space with a positive-definite hermitian inner product, and let $A\colon E \to E$ be a $\mathbb{C}$-linear map, with adjoint A^*.

(a) Show that there exists an orthonormal basis of E with respect to which the matrix of A is upper triangular (i.e., has only 0's below the diagonal.)

(b) Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the eigenvalues of A, each occurring in this sequence the number of times equal to its algebraic multiplicity. Prove that

$$\sum_{i=1}^{n} |\lambda_i|^2 \leq \text{trace of } A^*A$$

with equality if and only if A is normal (i.e., $A^*A = AA^*$).

4. (a) Let M be a finitely-generated torsion module over a principal ideal domain R, with elementary divisors $p_i^{e_i}$ $(1 \leq i \leq n)$, (the p_i being primes in R). Define an isomorphism from the ring of R-homomorphisms of M into itself to a ring of $n \times n$ matrices in which the (i,j) entry lies in $\text{Hom}_R(R/p_j^{e_j}, R/p_i^{e_i})$. (Just describe the map—don't give a detailed demonstration that it actually is an isomorphism.)

(b) Let A be an $n \times n$ matrix over a field k. Specify—with justification—the dimension of the vector space of $n \times n$ matrices (over k) which commute with A, in terms of the elementary divisors of the matrix $A - XI$ (where X is an indeterminate and I is the $n \times n$ identity matrix).