Math 554 **Qualifying Exam** **January 1994**

Name: ____

(18) 1. Let V be an n-dimensional vector space over an algebraically closed field F and let $\mathcal{F}$ be a commuting family of linear operators on V. Prove that there exists an ordered basis $\mathcal{B}$ for V such that every operator in $\mathcal{F}$ is represented by an upper triangular matrix with respect to $\mathcal{B}$.

Notation. For R a commutative ring and m and n positive integers, $R^{m\times n}$ denotes the set of $m \times n$ matrices over R.

(20) 2. Let A and B be $n \times n$ matrices over the field $\mathbb{Q}$ of rational numbers.

(i) Define "A and B are similar over $\mathbb{Q}$".

(ii) True or False: "if A and B are similar over the field $\mathbb{C}$ of complex numbers, then A and B are also similar over $\mathbb{Q}$". Justify your answer.

(iii) Let M and N be $n \times n$ matrices over the polynomial ring $\mathbb{Q}[t]$. Define "M and N are equivalent over $\mathbb{Q}[t]$".

(iv) True or False: "Every matrix $M \in \mathbb{Q}[t]^{n\times n}$ is equivalent to a diagonal matrix". Justify your answer.

(18) 2. (continued)

(v) True or False: "Every matrix $A \in \mathbb{Q}^{n\times n}$ is similar to a diagonal matrix over $\mathbb{C}$". Justify your answer.

(vi) Let $I \in \mathbb{Q}[t]^{n\times n}$ be the identity matrix. True or False: "if A and B are similar over $\mathbb{Q}$, then $tI - A$ and $tI - B$ are equivalent over $\mathbb{Q}[t]$". Justify your answer.

(vii) True or False: "if $tI - A$ and $tI - B$ are equivalent over $\mathbb{Q}[t]$, then A and B are similar over $\mathbb{Q}$". Justify your answer.

(18) 3. Let $\mathbb{Z}$ denote the ring of integers. Sketch a proof that if M is a $\mathbb{Z}$-submodule of the free $\mathbb{Z}$-module $\mathbb{Z}^n$, then M is a free $\mathbb{Z}$-module.

(15) 4. Let V be a finite-dimensional vector space over an algebraically closed field F and let $T : V \to V$ be a linear operator. True or False: "if $T = D + N$ where D is diagonalizable and N is nilpotent, then $D \circ N = N \circ D$". Justify your answer.

(15) 5. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear operator "left multiplication by $\begin{bmatrix} 2 & 0 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$".

True or False: "if W is a T-invariant subspace of $\mathbb{R}^3$, then there exists a T-invariant subspace W^1 of $\mathbb{R}^3$ such that $W \oplus W^1 = \mathbb{R}^3$." Justify your answer.

(16) 6. Let $F = \mathbb{F}_7$ be a finite field with 7 elements.

(i) What is the order of the multiplicative group $GL_2(\mathbb{F}_7)$ of 2×2 invertible matrices with entries from $\mathbb{F}_7$?

(ii) What is the order in $GL_2(\mathbb{F}_7)$ of the matrix $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$?

(iii) What is the order in $GL_2(\mathbb{F}_7)$ of the matrix $\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$?

(iv) What is the order of the group $SL_2(\mathbb{F}_7)$ of matrices in $GL_2(\mathbb{F}_7)$ having determinant 1?

(15) 7. Let F be a field and let $V = F^{4\times 4}$ be the vector space of 4×4 matrices over F. For $A \in F^{4\times 4}$, define $T_A : V \to V$ by $T_A(B) = AB$ for each $B \in V$.
True or False: "The minimal polynomial of T_A is never equal to the characteristic polynomial of T_A". Justify your answer.

(20) 8. Let F be a field, let m and n be positive integers and let $A \in F^{m \times n}$ be an $m \times n$ matrix.

(i) Define "row space of A".

(ii) Define "column space of A".

(iii) Sketch a proof that the dimension of the row space of A is equal to the dimension of the column space of A.

(15) 9. Let $\varphi : \mathbb{Z}^2 \to \mathbb{Z}^3$ be defined by left multiplication by the matrix $\begin{bmatrix} 2 & 6 \\ 4 & 8 \\ 10 & 10 \end{bmatrix}$ and let $V = \mathbb{Z}^3/\varphi(\mathbb{Z}^2)$. Decompose V as a direct sum of cyclic abelian groups.

(15) 10. Assume that $A \in \mathbb{R}^{3\times 3}$ has eigenvalues $0, 2, 4$ and that v_0, v_2, v_4 are associated eigenvectors.

(i) Give a basis for the column space of A.

(ii) Describe all solutions of the system of equations $AX = v_2 + v_4$.

(15) 11. Let t be an indeterminate over the field $\mathbb{R}$ and let $\varphi : \mathbb{R}[t]^3 \to \mathbb{R}[t]^3$ be defined by left multiplication by the matrix $\begin{bmatrix} t-1 & 0 & 0 \\ 2 & t-1 & 0 \\ 3 & 0 & t-1 \end{bmatrix}$. Decompose $\mathbb{R}[t]^3/\varphi(\mathbb{R}[t]^3)$ as a direct sum of cyclic $\mathbb{R}[t]$-modules.