Math 554 **Qualifying Exam** **August 1994**

Name: ____

Notation. If R is a commutative ring and m and n are positive integers, then $R^{m\times n}$ denotes the set of $m \times n$ matrices with entries in R.

(20) 1. Let F be an algebraically closed field. If $\mathcal{S}$ is a family of commuting matrices in $F^{3\times 3}$, prove that $\mathcal{S}$ does not contain 4 linearly independent matrices.

(20) 2. Let F be an algebraically closed field.

True or false? If $\mathcal{S}$ is a family of commuting matrices in $F^{4\times 4}$, then $\mathcal{S}$ does not contain 5 linearly independent matrices. Justify your answer.

(20) 3. Let F be a field and let n be a positive integer. Suppose that $T : F^n \to F^n$ is a linear operator having n distinct characteristic values and $S : F^n \to F^n$ is a linear operator that commutes with T. Prove that there exists a polynomial $f(t) \in F[t]$ such that $S = f(T)$.

(20) 4. Let $V = \mathbb{Z}^3$ and let L be the submodule of V spanned by the columns of

$$A = \begin{bmatrix} 4 & 2 & 2 \\ 2 & 2 & 6 \\ 2 & 4 & 4 \end{bmatrix}.$$

(i) Find Q and P in $\mathrm{GL}_3(\mathbb{Z})$ so that QAP is diagonal.

(ii) Find a basis $(\vec{\alpha}_1, \vec{\alpha}_2, \vec{\alpha}_3)$ of V and integers c_1, c_2, c_3 so that $(c_1\vec{\alpha}_1, c_2\vec{\alpha}_2, c_3\vec{\alpha}_3)$ is a basis for L.

(20) 5. Let V be a finite-dimensional vector space over a field F, let $T : V \to V$ be a linear operator, and let $m(t) \in F[t]$ be the minimal polynomial of T. Assume that $m(t) = p_1(t)^{e_1} \cdots p_k(t)^{e_k}$, where the $p_i(t) \in F[t]$ are distinct monic irreducible polynomials, $i = 1, \cdots, k$, and the e_i are positive integers. Let $W_i = \{v \in V : p_i(T)^{e_i}(v) = 0\}$.

(i) Describe how to get linear operators $E_i : V \to V$, $i = 1, \ldots, k$, such that $E_i(V) = W_i$, $E_i \circ E_i = E_i$, $E_i \circ E_j = 0$ if $i \neq j$, and $E_1 + \cdots + E_k$ is the identity operator on V.

(ii) If W is a T-invariant subspace of V, prove that $W = (W \cap W_1) \oplus \cdots \oplus (W \cap W_k)$.

(20) 6. Let $A \in \mathbb{C}^{3\times 3}$ be a diagonal matrix with main diagonal entries c_1, c_2, c_3, where c_1, c_2, c_3 are three distinct complex numbers. Define $T_A : \mathbb{C}^{3\times 3} \to \mathbb{C}^{3\times 3}$ by $T_A(B) = AB - BA$.

(i) What is $\dim(\ker(T_A))$?

(ii) What is $\dim(\operatorname{im}(T_A))$?

(iii) What are the eigenvalues of T_A?

(iv) What is the minimal polynomial of T_A?

(v) Is T_A diagonalizable? Explain.

(20) 7. Let $A \in \mathbb{C}^{3\times 3} = V$ and let $A' = PAP^{-1}$, where $P \in GL_3(\mathbb{C})$. Define $T_A : V \to V$ by $T_A(B) = AB - BA$ and $T_{A'} : V \to V$ by $T_{A'}(B) = A'B - BA'$. Let $M \in \mathbb{C}^{9\times 9}$ be the matrix representing T_A with respect to the ordered basis $(\vec{\alpha}_1, \ldots, \vec{\alpha}_9)$ of V.

(i) Does M represent $T_{A'}$ with respect to some ordered basis of V? Justify your answer.

(ii) Are the linear operators T_A and $T_{A'}$ similar? Explain.

(20) 8. Let $a_1 \geq a_2 \geq \cdots \geq a_r$ and $b_1 \geq b_2 \geq \cdots \geq b_s$ be positive integers and let

$$V = \mathbb{Z}/(5^{a_1}) \oplus \mathbb{Z}/(5^{a_2}) \oplus \cdots \oplus \mathbb{Z}/(5^{a_r})$$

and

$$W = \mathbb{Z}/(5^{b_1}) \oplus \mathbb{Z}/(5^{b_2}) \oplus \cdots \oplus \mathbb{Z}/(5^{b_s}).$$

If the abelian groups V and W are isomorphic, prove that $r = s$ and that $a_i = b_i, i = 1, \ldots, r$.

(20) 9. Let V be the $\mathbb{Z}$-module $\mathbb{Z}/(3) \oplus \mathbb{Z}/(3) \oplus \mathbb{Z}/(9)$.

(i) How many submodules with 3 elements does V have?

(ii) How many of the submodules W of V with 3 elements have a complementary direct summand, i.e., are such that there exists a submodule W' of V with $V = W \oplus W'$?

(iii) How many cyclic submodules with 9 elements does V have?

(iv) How many noncyclic submodules with 9 elements does V have?

(20) 10. Let V be a vector space, let $T : V \to V$ be a linear operator, and let k and n be positive integers.

(i) If $w \in V$ is such that $T^k(w) \neq 0$ and $T^{k+1}(w) = 0$, must $(w, T(w), \cdots, T^k(w))$ be linear independent? Explain.

(ii) Let W be the subspace of V spanned by $(w, T(w), \cdots, T^k(w))$ as in part (i). If $v \in V$ is such that $T^n(v) \notin W$ and $T^{n+1}(v) \in W$, must $(w, T(w), \cdots, T^k(w), v, T(v), \cdots, T^n(v))$ be linearly independent? Explain.