QUALIFYING EXAM COVER SHEET

January 2020 Qualifying Exams

Instructions: These exams will be “blind-graded” so under the student ID number please use your PUID

ID #: ____________________________
(10 digit PUID)

EXAM (circle one) 514 519 523 530 544 **553** 554 562 571

For grader use:

Points _________ / Max Possible_________ Grade _________

Instructions:

1. The point value of each exercise occurs adjacent to the problem.

2. No books or notes or calculators are allowed.

Page	**Points Possible**	**Points**
2	20	
3	20	
4	20	
5	20	
6	20	
7	20	
8	20	
9	20	
10	20	
11	20	
Total	200	

1. (8 pts) Let $g(x) \in F[x]$ be an irreducible monic polynomial of $\deg n \geq 1$, where F is a field of characteristic zero. Prove that $g(x)$ does not have a multiple root.

2. (6 pts) Give an example of a field F having characteristic $p > 0$ and an irreducible monic polynomial $g(x) \in F[x]$ that has a multiple root.

3. (6 pts) Give an example of a finite algebraic field extension L/F such that there exist infinitely many fields K with $F \subseteq K \subseteq L$.

4. (5 pts) State Zorn's Lemma.

5. (15 pts) Let R be a commutative ring with $1 \neq 0$. Assume that $a \in R$ is such that $a^n \neq 0$ for each positive integer n, and let $\mathcal{S} = \{a^n\}_{n \geq 0}$.

(i) Prove that there exists an ideal I of R such that I is maximal among ideals of R with $I \cap \mathcal{S} = \emptyset$.

(ii) Prove that an ideal I as in item (i) is a prime ideal.

(iii) Give an example of a ring R having an element a such that a is a zero divisor and $a^n \neq 0$ for each positive integer n.

6. (20 pts) Define what is meant by a composition series for a finite group G.

(a) State the Jordan-Hölder Theorem.

(b) Diagram the lattice of subgroups of the symmetric group S_3 and exhibit all the composition series for S_3. How many are there?

(c) Diagram the lattice of subgroups of the quaternion group Q_8 and exhibit all the composition series for Q_8. How many are there?

(d) How many composition series exist for the dihedral group D_8? Justify your answer.

7. (20) Let p be a prime number, and let $\mathbb{F}_p$ denote the finite field with p elements.

(i) Prove that every finite algebraic extension field of $\mathbb{F}_p$ is Galois.

(ii) Let K and L be finite algebraic field extensions of $\mathbb{F}_p$.

(a) If $[K : \mathbb{F}_p] = [L : \mathbb{F}_p]$, does it follow that K is isomorphic to L? Justify your answer.

(b) If $[K : \mathbb{F}_p] \leq [L : \mathbb{F}_p]$, does it follow that K is isomorphic to a subfield of L? Justify your answer.

(iii) Let $\overline{\mathbb{F}_p}$ denote the algebraic closure of $\mathbb{F}_p$. If E is a subfield of $\overline{\mathbb{F}_p}$ and $[E : \mathbb{F}_p] = \infty$, does it follow that E is algebraically closed? Justify your answer.

8. Let n and p be positive integers with p a prime integer. Let $Z = \langle x \rangle$ be a cyclic group of order $p^n - 1$.

(a) (7 pts) Describe the group $\mathrm{Aut}(Z)$ of automorphism of Z. In particular, what is $|\mathrm{Aut}(Z)|$?

(b) (7 pts) Let $\mathbb{F}_p$ be the field with p elements and let $L/\mathbb{F}_p$ be a field extension of degree n. Let G be the Galois group of $L/\mathbb{F}_p$. Describe the group G. In particular, what is $|G|$?

9. (6 pts) Let G be a finite group and let C be the center of G. If G/C is abelian, does it follow that $C = G$? Justify your answer.

10. Let L/F be a finite algebraic field extension.

(a) (10) If $L = F(\alpha)$ for some $\alpha \in L$, prove that there are only finitely many subfields K of L with $F \subseteq K$.

(b) (10) If there are only finitely many subfields K of L with $F \subseteq K$, prove that there exists an element $\alpha \in L$ such that $L = F(\alpha)$.

11. (10 pts) Let F be a field and let $F(x)$ denote the field of fractions of the polynomial ring $F[x]$. Let $\operatorname{Aut} F(x)$ denote the group of automorphisms of the field $F(x)$, and let $\sigma \in \operatorname{Aut} F(x)$ be such that σ fixes F and $\sigma x = x + 1$. Let $G = \langle \sigma \rangle$ be the cyclic subgroup of $\operatorname{Aut} F(x)$ generated by σ.

(a) Depending on the characteristic of the field F, what is the order of the group G?

(b) Depending on the characteristic of the field F, give generators for the fixed field $F(x)^G$.

12. (10 pts) Let p be a prime number and let $K/\mathbb{Q}$ be a splitting field of the polynomial $f(x) = x^p - 2 \in \mathbb{Q}[x]$.

(a) What is the degree of K over $\mathbb{Q}$?

(b) Give generators for K over $\mathbb{Q}$.

13. (10 pts) Let $\alpha = \sqrt{2 + \sqrt{2}} \in \mathbb{R}$.

(a) What is the minimal polynomial for α over $\mathbb{Q}$?

(b) List the conjugates of α over $\mathbb{Q}$.

(c) List the conjugates of α over $\mathbb{Q}(\sqrt{2})$.

(d) Prove that $\mathbb{Q}(\alpha)/\mathbb{Q}$ is Galois and describe the group $\mathrm{Aut}(\mathbb{Q}(\alpha)/\mathbb{Q})$.

14. (10 pts) Let $\beta = \sqrt{1 + \sqrt{3}} \in \mathbb{R}$.

(a) What is the minimal polynomial for β over $\mathbb{Q}$?

(b) List the conjugates of β over $\mathbb{Q}$.

(c) List the conjugates of β over $\mathbb{Q}(\sqrt{3})$.

(d) Is $\mathbb{Q}(\beta)/\mathbb{Q}(\sqrt{3})$ Galois ?

(e) Let K be the splitting field of the minimal polynomial of β over $\mathbb{Q}$. What is $[K : \mathbb{Q}]$?

15. Let $K/\mathbb{Q}$ be the splitting field of the polynomial $x^4 + 1 \in \mathbb{Q}[x]$.

(a) (4 pts) What is the degree $[K : \mathbb{Q}]$?

(b) (8 pts) If α is one root of x^4+1, diagram the lattice of fields between $\mathbb{Q}$ and $\mathbb{Q}(\alpha)$, and give generators for each intermediate field.

16. (8 pts) True or false: If $f(x), g(x) \in \mathbb{Q}[x]$ are irreducible polynomials that have the same splitting field, then $\deg f = \deg g$. Justify your answer.

17. (20) Let $n > 1$ be a positive integer and let p be a prime integer. Let $\varphi : \frac{\mathbb{Z}}{(pn)} \to \frac{\mathbb{Z}}{(n)}$ be the natural surjective ring homomorphism.

(a) If p does not divide n and x is a unit in $\frac{\mathbb{Z}}{(n)}$, is every element in $\varphi^{-1}(x)$ a unit in $\frac{\mathbb{Z}}{(pn)}$? Justify your answer.

(b) If p divides n and x is a unit in $\frac{\mathbb{Z}}{(n)}$, is every element in $\varphi^{-1}(x)$ a unit in $\frac{\mathbb{Z}}{(pn)}$? Justify your answer.

(c) Prove that φ maps the units of $\frac{\mathbb{Z}}{(pn)}$ surjectively onto the units of $\frac{\mathbb{Z}}{(n)}$,