Math 553 **Qualifying Exam** **August 10, 2020** **W. Heinzer**

Instructions:

1. The point value of each exercise occurs adjacent to the problem.

2. No books or notes or calculators are allowed.

Page	**Points Possible**	**Points**
2	20	
3	20	
4	20	
5	20	
6	20	
7	20	
8	20	
9	20	
10	20	
11	20	
Total	200	

1. For each of the following state true or false. If the statement is true describe as best you can how the result is proved. On the other hand, if the statement is false, explain why it is false, and if possible give an example to show the statement is false

(a) (5 pts) Let F be a field. If $g(x) \in F[x]$ is an irreducible monic polynomial of $\deg n \geq 1$, then $g(x)$ does not have a multiple root.

(b) (5 pts) If F is a finite field and K/F is a finite algebraic extension, then K/F is a simple extension.

(c) (5 pts) If G is a finite group and every subgroup of G is normal in G, then G is an abelian group.

(d) (5 pts) If R is an integral domain in which every prime ideal is a principal ideal, then R is a principal ideal domain.

2. Continue as on the previous page.

(a) (5 pts) If R is a unique factorization domain, then every nonzero prime ideal of R is a maximal ideal.

(b) (5 pts) If $f(x), g(x) \in \mathbb{Q}[x]$ are irreducible polynomials that have the same splitting field, then $\deg f = \deg g$.

(c) (5 pts) If G is an abelian group and $H \leq G$ is a proper subgroup, then there exists a maximal subgroup M of G such that $H \leq M$.

(d) (5 pts) In the quadratic integer ring $R = \mathbb{Z}[\sqrt{-5}]$, every nonzero prime ideal is a maximal ideal.

3. Continue as on the previous page.

(a) (5 pts) If L/F is a finite algebraic field extention, then there exist only finitely many fields K such that $F \subseteq K \subseteq L$.

(b) The ring $R = \mathbb{Z}[i]$ of Gaussian integers has infinitely many maximal ideals.

(c) If G is a finite simple group, then G is a solvable group.

(d) If G is a finitely generated group that has a unique maximal subgroup, then G is a finite cyclic group.

4. (5 pts) State Zorn's Lemma.

5. (15 pts) Let R be a commutative ring with $1 \neq 0$. Assume that $a \in R$ is such that $a^n \neq 0$ for each positive integer n, and let $\mathcal{S} = \{a^n\}_{n \geq 0}$.

(i) Prove that there exists an ideal I of R such that I is maximal among ideals of R with $I \cap \mathcal{S} = \emptyset$.

(ii) Prove that an ideal I as in item (i) is a prime ideal.

(iii) Give an example of a ring R having an element a such that a is a zero divisor and $a^n \neq 0$ for each positive integer n.

6. (5 pts) Let G be a finite group with $|G| > 1$. Define a *composition series* for G.

(a) (5 pts) State the Jordan-Hölder Theorem.

(b) (10 pts) Diagram the lattice of subgroups of the alternating group A_4 and exhibit all the composition series for A_4. How many composition series are there?

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7. (20) Let L/F be a Galois field extension and let $G = \text{Aut}(L/F)$ be the Galois group of L/F.

(a) Describe the correspondence between fields K with $F \subseteq K \subseteq L$ and subgroups H of G given by the fundamental theorem of Galois theory.

(b) If G is the alternating group A_4, diagram the lattice of fields K with $F \subseteq K \subseteq L$. For each K, list the integers $[L : K]$ and $[K : F]$.

8. (20) Let p be a prime number, and let $\mathbb{F}_p$ denote the finite field with p elements.

(i) Prove that every finite algebraic extension field of $\mathbb{F}_p$ is Galois.

(ii) Let K and L be finite algebraic field extensions of $\mathbb{F}_p$.

(a) If $[K : \mathbb{F}_p] = [L : \mathbb{F}_p]$, does it follow that K is isomorphic to L? Justify your answer.

(b) If $[K : \mathbb{F}_p] \leq [L : \mathbb{F}_p]$, does it follow that K is isomorphic to a subfield of L? Justify your answer.

(iii) Let $\overline{\mathbb{F}_p}$ denote the algebraic closure of $\mathbb{F}_p$. If E is a subfield of $\overline{\mathbb{F}_p}$ and $[E : \mathbb{F}_p] = \infty$, does it follow that E is algebraically closed? Justify your answer.

9. Let n and p be positive integers with p a prime integer. Let $Z = \langle x \rangle$ be a cyclic group of order $p^n - 1$.

(a) (7 pts) Describe the group $\mathrm{Aut}(Z)$ of automorphism of Z. In particular, what is $|\mathrm{Aut}(Z)|$?

(b) (7 pts) Let $\mathbb{F}_p$ be the field with p elements and let $L/\mathbb{F}_p$ be a field extension of degree n. Let G be the Galois group of $L/\mathbb{F}_p$. Describe the group G. In particular, what is $|G|$?

(c) (6 pts) Prove that n divides $\phi(p^n - 1)$, where ϕ is the Euler ϕ-function.

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10. (10) Let L/F be a finite algebraic field extension If $L = F(\alpha)$ for some $\alpha \in L$, prove that there are only finitely many subfields K of L with $F \subseteq K$.

11. (10). Let L/F be a finite algebraic field extension, where F is an infinite field. If there are only finitely many subfields K of L with $F \subseteq K$, prove that there exists an element $\alpha \in L$ such that $L = F(\alpha)$.

12. (10 pts) Let F be a field and let $F(x)$ denote the field of fractions of the polynomial ring $F[x]$. Let $\operatorname{Aut} F(x)$ denote the group of automorphisms of the field $F(x)$, and let $\sigma \in \operatorname{Aut} F(x)$ be such that σ fixes F and $\sigma x = x + 1$. Let $G = \langle \sigma \rangle$ be the cyclic subgroup of $\operatorname{Aut} F(x)$ generated by σ.

(a) Depending on the characteristic of the field F, what is the order of the group G?

(b) Depending on the characteristic of the field F, give generators for the fixed field $F(x)^G$.

13. (10 pts) Let p be a prime number and let $K/\mathbb{Q}$ be a splitting field of the polynomial $f(x) = x^p - 2 \in \mathbb{Q}[x]$.

(a) What is the degree of K over $\mathbb{Q}$?

(b) Give generators for K over $\mathbb{Q}$.