Qualifying Examination
MA 553
January 2, 2019
Time: 2 hours

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Your ID: ______________________________

(15 pts) 1). Show that every group of order $4125 = 3 \cdot 5^3 \cdot 11$ is solvable.

Your ID: ______________________________

(15 pts) 2). Let α be a root of

$$f(x) = x^{19} - 7x^{16} + 77x^{11} + 63x^5 - 35x^3 + 14 = 0.$$

Is $\mathbb{Q}(\alpha^8) = \mathbb{Q}(\alpha)$? Justify your answer.

Your ID: ______________________

(20 pts) 3). Let R be a commutative ring with $1 \neq 0$. Show that the following are equivalent:

a) R has a unique maximal ideal M.

b) The set $R \setminus R^*$ of the non-units is an ideal.

Your ID: ______________________________

(20 pts) 4). Let R be a commutative ring with $1 \neq 0$ and let P be a prime ideal of R. Let I and J be ideals of R such that $I \cap J \subset P$. Prove that either $I \subseteq P$ or $J \subseteq P$.

Your ID: ________________________________

5).

(30 pts) a) Let p and q be odd primes with $q \equiv 1(4)$. Let α be a root of $x^4 + p \in \mathbb{F}_q[x]$, where $\mathbb{F}_q := \mathbb{Z}/q\mathbb{Z}$. Assume q is not a square in $\mathbb{F}_p$. Show that $[\mathbb{F}_q(\alpha) : \mathbb{F}_q] = 4$.

(Hint: Use quadratic reciprocity law $(\frac{p}{q})(\frac{q}{p}) = (-1)^{\frac{p-1}{2} \cdot \frac{q-1}{2}}$, p and q odd primes, where $(\frac{p}{q}) = 1$ or -1 according as the equation $x^2 \equiv p (\bmod q)$ has a solution or not.)

(5 pts) b) Conclude that $\mathbb{F}_q(\alpha) \simeq \mathbb{F}_{q^4}$ and thus a Galois extension of $\mathbb{F}_q$.

Your ID: ______________________________

Your ID: ________________________________

6). Show that the following polynomials are irreducible.

(10 pts) a) $x^3 + 4 \in \mathbb{Q}[x]$.

(10 pts) b) $x^2 + y^n + y$ in $\mathbb{Z}[x, y]$.

Your ID: ______________________________

7). Find a simpler description for each of the following rings.

(20 pts) a) $\mathbb{Q}[x]/(x^5 + x^3)$.

(20 pts) b) $\mathbb{Z}[x]/(x - 2, x^2 + 3x)$.

Your ID: ______________________________

8).

(7 pts) a) Show that $f(x) = x^6 - 2x^3 - 10$ is irreducible over $\mathbb{Q}$.

(5 pts) b) Let $\alpha = \sqrt[3]{1+\sqrt{11}}$. Show that $[\mathbb{Q}(\alpha) : \mathbb{Q}] = 6$.

(10 pts) c) Show that $K = \mathbb{Q}(\alpha, \sqrt{-3}, \sqrt[3]{10})$ is a splitting field for $f(x)$.

(8 pts) d) Show that $\sqrt[3]{10} \notin \mathbb{Q}(\sqrt{-3}, \sqrt{11})$ and conclude that $[L : \mathbb{Q}] = 12$, where $L = \mathbb{Q}(\sqrt{-3}, \sqrt{11}, \sqrt[3]{10})$. Moreover $K = L(\alpha)$ and thus $[K : \mathbb{Q}] = 12$ or 36.

(5 pts) e) Show that $K/\mathbb{Q}$ is an extension by radicals. Is $K/\mathbb{Q}$ solvable? Justify your answer.

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