Qualifying Examination
MA 553
August 8, 2019
Time: 2 hours

Your ID: ______________________________

1	
2	
3	
4	
5	
6	
7	
Total	

In answering any part of a question, you may assume the results in previous parts, even if you have not solved them. Be sure to provide all details of your work.

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(25 pts) 1). Show that the alternating subgroup A_{11} of S_{11} cannot have a subgroup of order $2,851,200 = 2^7 \cdot 3^4 \cdot 5^2 \cdot 11$.

(Hint: A_{11} is simple. You need not prove this.)

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(10 pts) 2). Show that any group of order 294 is solvable.

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3).

(10 pts) a) Let R be a non-zero commutative ring with 1. Show that if I is an ideal of R such that $1+a$ is a unit in R for all $a \in I$, then I is contained in every maximal ideal of R.

(20 pts) b) Let $m \subset R$ be a unique maximal ideal. Then $a \in m$ if and only if $1 + ca$ is a unit.

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4). Let $\phi : R \to S$ be a surjective homomorphism of commutative rings with $1 \neq 0$ and assume that R contains a unique maximal ideal.

(10 pts) a) Show that S contains a unique maximal ideal.

(10 pts) b) $\phi(1_R) = 1_S$.

(10 pts) c) Show that an element is a unit in S if and only if it is the image of a certain unit in R.

(10 pts) d) Show that (b) is not true if $\phi : R \to S$ is not surjective.

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5). Show that

(10 pts) a) $(x^2 + y^2)$ is irreducible in $\mathbb{R}[x, y]$.

(10 pts) b) $\mathbb{R}[x, y]/(x^2 + y^2)$ is an integral domain.

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6). Determine the Galois groups of the following polynomials:

(10 pts) a) $f(x) = x^3 - 2x + 4$.

(10 pts) b) $g(x) = x^3 - 3x + 1$.

(10 pts) c) $h(x) = f(x)g(x)$.

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7). Let $\alpha = \sqrt[3]{1+\sqrt{3}}$ and $\beta = \sqrt[3]{1-\sqrt{3}}$

(5 pts) a) Show that $[\mathbb{Q}(\alpha) : \mathbb{Q}] = 6$

(10 pts) b) Prove that $K = \mathbb{Q}(\alpha, \beta, \sqrt{-1})$ is a normal closure of $\mathbb{Q}(\alpha)/\mathbb{Q}$.

(10 pts) c) Show that $\sqrt[3]{2} \in K$.

(10 pts) d) Show that $\sqrt[3]{2} \notin \mathbb{Q}(\sqrt{3}, \sqrt{-1})$, but $\omega \in \mathbb{Q}(\sqrt{3}, \sqrt{-1})$, where ω is a root of $\omega^2+\omega+1 = 0$. Conclude that $[L : \mathbb{Q}] = 12$, where $L = \mathbb{Q}(\sqrt[3]{2}, \sqrt{3}, \sqrt{-1})$. Thus $[K : \mathbb{Q}] = 12$ or 36.

(10 pts) e) Show that both $K/\mathbb{Q}$ and $L/\mathbb{Q}$ are extensions by radicals. Is $K/\mathbb{Q}$ solvable? Justify your answer.