FALL 2016

Qualifying Exam - MA 553

In answering any part of a question, you may assume the results in previous parts, even if you have not solved them. Be sure to provide all details of your work.

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1. Let $p < q$ be prime numbers, and G be any group of order p^2q.
   (a) (5 points) Show that the number n_q of Sylow q-subgroups in G is either 1 or p^2.
   (b) (5 points) Show that if $n_q = p^2$ then G contains a unique Sylow p-subgroup.
   (c) (10 points) Show that G is solvable.

2. (20 points) Let $H \subset G$ be a normal subgroup of G. Suppose that P is a Sylow p- subgroup of H. Show that the following conditions are equivalent

   (a) P is normal in G

   (b) P is normal H

3. Let G be a group of order 66.

   (a) (5 point) Show that G has a normal subgroup of order 11.

   (b) (5 points) Show that G has a normal subgroup of order 33.

   (c) (5 points) Show that G has an element of order 33.

   (d) (5 points) Show that G cannot be embedded in S_{12}.

4. (20 points) Let $(R, +,)$ be a commutative ring with $1 \neq 0$ and assume that $M \subset R$ is a maximal ideal. Show that the following conditions are equivalent:

(a) $x \notin M$.

(b) There exists $a \in R$ such that $1 + ax \in M$.

5. (20 points) Let R be a commutative ring with $1 \neq 0$ and let P be a prime ideal of R. Let I and J are ideals of R such that $I \cap J \subset P$, Prove that either $I \subseteq P$ or $J \subseteq P$.

6. (a) Prove that $Z[x]$ is not PID.(10 pts).

   (b) Show that $Z[x]/(x^2 + 5)$ is not UFD. (10 points).

7. Let L be the splitting field of a polynomial $f(x) \in F[x]$ of degree n over a field F.

   (a) Show that $[L : F] \leq n!$ (10 points)

   (b) Show that n divides $[L : F]$ whenever $f(x)$ is irreducible. (10 points)

8. Consider the polynomial $f(x) = (x^6 + 1)(x^4 + 1)(x^3 + x^2 + 1)$ over F_2.

(a) (10 points) Find the splitting field L of f over F_2.

(b) (5 points) Find the Galois group of L , and its generators.

(c) (5 points) Describe all intermiediate fields $F_2 \subseteq K \subseteq L$.

9. Let ϵ_n be a primitive n-th root of unity for some natural n.

(a) (5 points) Show that $Q(\epsilon_n)/Q$ is Galois.

(b) (5 points) Describe its Galois group.

(c) (10 points) Show that $Q(\epsilon_n)$ does not contain $\sqrt[3]{5}$.

10. Let L/Q be the splitting field of the polynomial $x^6 - 2 \in Q[x]$.

(a) (5 points) What is the degree $[L : Q]$?

(b) (5 points) What is the Galois group $Gal(L/Q)$.

(c) (5 points) Give generators for each subfield K of L for which $[K : Q] = 2$. How many K are there?

(d) (5 points) Find the number of the subfields K of L for which $[K : Q] = 4$.