Math 553 **Qualifying Exam** **August, 2014** **W. Heinzer**

Instructions:

1. The point value of each exercise occurs adjacent to the problem.

2. No books or notes or calculators are allowed.

Page	**Points Possible**	**Points**
2	20	
3	20	
4	18	
5	18	
6	24	
7	18	
8	24	
9	20	
10	18	
11	20	
Total	200	

1. Let G be a finite group and H a subgroup such that $|G : H| = d$ with $1 < d < |G|$.

(a) (5 pts) Describe the natural homomorphism $\phi : G \to S_d$, where S_d is the permutation group on the left cosets of H in G.

(b) (5 pts) If $|G| = n$ and d is the smallest prime dividing n, prove that H is normal in G.

(c) (5 pts) If $|G| = 24$ and $d = 3$, prove that G contains a normal subgroup of order 4 or 8.

(d) (5 pts) If $|G| = 24$ and $d = 3$, must H be normal in G? Justify your answer.

2. A sequence of subgroups $1 = N_0 \leq N_1 \leq \cdots \leq N_{k-1} \leq N_k = G$ is called a *composition series* for a group G if N_i is normal in N_{i+1} and N_{i+1}/N_i is a simple group for $0 \leq i \leq k-1$.

(a) (5 pts) State the Jordan-Hölder Theorem for a finite group.

(b) (5 pts) Diagram the lattice of subgroups of the symmetric group S_3 and exhibit all the composition series for S_3. How many are there?

(c) (5 pts) Diagram the lattice of subgroups of the quaternion group Q_8 and exhibit all the composition series for Q_8. How many are there?

(d) (5 pts) How many composition series exist for the dihedral group D_8? Justify your answer.

3. (8 pts) Let K/F be an algebraic field extension. If $K = F(\alpha)$ for some $\alpha \in K$, prove that there are only finitely many subfields of K that contain F.

4. (10 pts) Let x and y be indeterminates over the field $\mathbb{F}_2$. Prove that there exist infinitely many subfields of $L = \mathbb{F}_2(x, y)$ that contain the field $K = \mathbb{F}_2(x^2, y^2)$.

5. Let p be a prime integer. Recall that a field extension K/F is called a p-extension if K/F is Galois and $[K : F]$ is a power of p.

(a) (10 pts) If K/F and L/K are p-extensions, prove that the Galois closure of L/F is a p-extension.

(b) (8 pts) Give an example where K/F and L/K are p-extensions, but L/F is not Galois.

6. Let $L/\mathbb{Q}$ be the splitting field of the polynomial $x^6 - 2 \in \mathbb{Q}[x]$.

(a) (4 pts) What is the degree $[L : \mathbb{Q}]$?

(b) (4 pts) If α is one root of $x^6 - 2$, diagram the lattice of fields between $\mathbb{Q}$ and $\mathbb{Q}(\alpha)$.

(c) (4 pts) Give generators for each subfield K of L for which $[K : \mathbb{Q}] = 2$. How many K are there?

(d) (4 pts) Give generators for each subfield K of L for which $[K : \mathbb{Q}] = 3$. How many K are there?

(e) (4 pts) Give generators for each subfield K of L for which $[K : \mathbb{Q}] = 4$. How many K are there?

(f) (4 pts) How many subfields K of L are such that $[L : K] = 2$?

7. (9 pts) Prove that an irreducible monic polynomial $f(x) \in \mathbb{Q}[x]$ cannot have a multiple root.

8. (9 pts) Give an example of a field F having characteristic $p > 0$ and an irreducible monic polynomial $f(x) \in F[x]$ that has a multiple root.

9. Consider the quadratic integer ring $R = \mathbb{Z}[\sqrt{-5}] \;=\; \frac{\mathbb{Z}[x]}{(x^2+5)\mathbb{Z}[x]}$.

(a) (6 pts) Give generators for each ideal of R that contains (3) and diagram the lattice of ideals of R that contain (3).

(b) (6 pts) Give generators for each ideal of R that contains (7) and diagram the lattice of ideals of R that contain (7).

(c) (6 pts) Including the whole ring R and the ideal (21), how many ideals of R contain the ideal (21)?

(d) (6 pts) Diagram the lattice of ideals of R that contain (21).

10. (10 pts) Let n be a positive integer and d a positive integer that divides n. Suppose $\alpha \in \mathbb{R}$ is a root of the polynomial $x^n - 2 \in \mathbb{Q}[x]$. Prove that there is precisely one subfield F of $\mathbb{Q}(\alpha)$ with $[F : \mathbb{Q}] = d$.

11. (10 pts) Does there exist an example of an infinite abelian group G such that every proper subgroup of G is a finite group? Justify your answer with either a proof or an example.

12. Let R be the polynomial ring $\mathbb{Z}[x]$

(a) (9 pts) How many maximal ideals of R contain the ideal $I = (15, x^2 + 2)R$? Give generators for each of these maximal ideals.

(b) (9 pts) Diagram the lattice of ideals of R that contain the ideal $I = (15, x^2 + 2)R$, giving generators for each ideal.

13. Let n and p be positive integers with p a prime integer. Let $Z = \langle x \rangle$ be a cyclic group of order $p^n - 1$.

(a) (7 pts) Describe the group $\mathrm{Aut}(Z)$ of automorphism of Z. In particular, what is $|\mathrm{Aut}(Z)|$?

(b) (7 pts) Let $\mathbb{F}_p$ be the field with p elements and let $L/\mathbb{F}_p$ be a field extension of degree n. Let G be the Galois group of $L/\mathbb{F}_p$. Describe the group G. In particular, what is $|G|$?

14. (6 pts) Let G be a finite group and let C be the center of G. If G/C is abelian, does it follow that $C = G$? Justify your answer.