ABSTRACT ALGEBRA COMPREHENSIVE EXAM – AUG, 2013

Attempt all questions. Time 2 hrs

(1) (20 pts) Let G be a group such that for a fixed integer $n > 1$, $(xy)^n = x^n y^n$ for all $x, y \in G$. Let $G^{(n)} = \{x^n | x \in G\}$ and $G_{(n)} = \{x \in G | x^n = e\}$.
   (a) Prove that $G^{(n)}$ and $G_{(n)}$ are normal subgroups of G.
   (b) If G is finite, show that the order of $G^{(n)}$ is equal to the index of $G_{(n)}$.
   (c) Show that for all $x, y \in G$, we have $x^{1-n} y^{1-n} = (xy)^{1-n}$. Use this to deduce that $x^{n-1} y^n = y^n x^{n-1}$.
   (d) Conclude from the above that the set of elements of G of the form $x^{n(n-1)}$ generates a commutative subgroup of G.

(2) (20 pts) Let G be a finite group and p a prime number. An element $g \in G$ is called *p-unipotent* if its order is a power of p, and *p-regular* if its order is not divisible by p.
   (a) Let $x \in G$. Show that there exists a unique ordered pair (u, r) of elements of G such that u is p-unipotent, r is p-regular, and $x = ur = ru$.
   (b) Let P be a p-Sylow subgroup of G, C the centralizer of P, and E the set of p-regular elements of G. Show that

$$|E| \equiv |E \cap C| \mod p.$$

   (c) Deduce that p does not divide the order of E.

(3) (10pts) G be a finite group, and $\varphi : G \longrightarrow G$ a homomorphism.
   (a) Prove that there exists a positive integer n such that for all integers $m \geq n$, we have $\text{Im}\varphi^m = \text{Im}\varphi^n$ and $\text{Ker}\varphi^m = \text{Ker}\varphi^n$.
   (b) For n as above, prove that G is the semidirect product of the subgroups $\text{Ker}\varphi^n$ and $\text{Im}\varphi^n$.

(4) (20 pts) In which of the following rings is every ideal principal? Justify your answer.

$$(i)\ \mathbb{Z} \oplus \mathbb{Z}, \qquad (ii)\ \frac{\mathbb{Z}}{(4)}, \qquad (iii)\ \frac{\mathbb{Z}}{(6)}[x], \qquad (iv)\ \frac{\mathbb{Z}}{(4)}[x].$$

(5) (20 pts) Let k be a field of characteristic $\neq 2, 3$. Prove that the following statements are equivalent:
   (a) Any sum of squares in k is itself a square.
   (b) Whenever a cubic polynomial f factors completely in k, so does its derivative f'.

(6) (10 pts)
   (a) Let k be a field and α algebraic over k and such that $[k(\alpha) : k]$ is odd. Prove that $k(\alpha^2) = k(\alpha)$.
   (b) Calculate $[E : \mathbb{Q}]$ where $E = \mathbb{Q}(\sqrt{2}, \sqrt{5}, \sqrt{7})$?
   (c) Is $2^{1/3}$ (the real cube root of 2) contained in E ? Justify.