**MATH 553 Qualifying Exam** January 2011

*Note.* There are 5 questions in 15 parts. Each part worths 8 points. But the maxmal points you may receive is 100 points. You may do them in any order but label your solutions clearly. You may do any problem/part by assuming the conclusion of other problems/parts.

**1.** Let $p$ be a prime and let $G = \mu_p \times \mu_p,$, where $\mu_p = \{\zeta \in \mathbb{C} : \zeta^p = 1\}$. Let $k$ be an extension of $\mathbb{Q}(\mu_p)$ and let $K = k(\alpha, \beta)$ be a finite extension of degree $p^2$ over $k$ such that $\alpha^p, \beta^p \in k$.

(a) Show that $G$ has exactly $p+1$ subgroups of order $p$.

(b) Show that $\mathrm{Gal}(K/k) \to G, \sigma \mapsto (\sigma(\alpha)/\alpha, \sigma(\beta)/\beta)$ is an isomorphism of groups.

(c) Let $E$ be a subfield of $K$ such that $k \subsetneq E \subsetneq K$. Show that $E$ is equal to one of

$$k(\alpha), k(\alpha\beta), \ldots, k(\alpha\beta^{p-1}), k(\beta).$$

**2.** Let $n$ be an integer and let $f(X) = X^3 - nX^2 - (n+3)X - 1$.

(a) Show that $f(X)$ is irreducible in $\mathbb{Q}[X]$.

(b) Show that if $\alpha$ is a root of $f(X)$, then $-1/(1+\alpha)$ is also a root of $f(X)$.

(c) Let $K$ be the splitting field of $f(X)$ over $\mathbb{Q}$. Show that $\mathrm{Gal}(K/\mathbb{Q})$ is cyclic of order 3.

**3.** Let $p$ be a prime number. Let $G$ be a finite subgroup of $S_p$, such that the order $n$ of $G$ is divisible by $p$, and $n < p^2$.

(a) Show that $G$ contains a $p$-cycle $c$.

(b) Show that the subgroup $N$ generated by $c$ is normal in $G$.

(c) Show that $G/N$ is cyclic. You may use the fact that $\mathbb{F}_p^\times$ is cyclic.

(d) Let $k$ be a field and $f \in k[X]$ be irreducible of degree $p$. Suppose that the splitting field $K$ of $f(X)$ over $k$ is generated by two roots of $f$. Show that $\mathrm{Gal}(K/k)$ has order $m < p^2$, and $m$ is divisible by $p$.

(e) Maintain the notaiton and hypothesis of (d). Show that $K/k$ contains a unique subextension of degree $m/p$ over $k$.

**4.** Let $G$ be a finite group and let be $P$ be a $p$-Sylow subgroup of $G$ for some prime $p$

(a) Assume that $P$ is cyclic and $p = 2$. Show that $N_G(P) = Z_G(P)$, where $N_G(P) = \{g \in G : gPg^{-1} = P\}$, $Z_G(P) = \{g \in G : gxg^{-1} = x \text{ for all } x \in P\}$.

(b) Show that $N_G(P)$ may be different from $Z_G(P)$ if $p = 2$ but $P$ is not cyclic by giving an example.

(b) Show that $N_G(P)$ may be different from $Z_G(P)$ if $p \neq 2$ by giving an example.

**5.** Consider the ring $R = \mathbb{Z}[X]/I$, where $I$ is the ideal of $\mathbb{Z}[X]$ generated by $X^3 + X + 1$, $X^3 + X - 1$. Determine the cardinality of $R$ and the structure of $R^\times$.