Qualifying Examination

MA 553

Time: 2 hours

August 13, 2010

Instructor: F. Shahidi

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(20 pts) 1. Show that every group of order 143 is cyclic. You are only allowed to use main theorems!

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2. Let p a prime.

(10 pts) a) Show that every group of order p^2 is abelian.

(10 pts) b) Show that every non–abelian group of order p^3 has a center of order p.

(5 pts) c) Using parts a) and b) show that every group of order p^3 is solvable. You CANNOT refer to any theorems except the definition of a solvable group by means of normal series.

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3.

(10 pts) a) Let R be a UFD (unique factorization domain) and consider

$$f(x,y) = x^5 + yx^3 + yx^2 + yx + y \in R[x,y].$$

Show that $f(x,y)$ is irreducible in $R[x,y]$.

(10 pts) b) Let $K = F(x^5/x^3 + x^2 + x + 1)$, where F is a field. Determine $[F(x) \colon K]$.

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4.

(5 pts) a) Give the definition of a euclidean domain.

(25 pts) b) Let A be the subring of all the complex numbers $a + b\sqrt{-7}$ in which a and b are both integers or both halves of integers. Prove that A is a euclidean domain. Is A a principal ideal domain (PID)? Prove this. Quoting a theorem is not acceptable.

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(15 pts) 5. Let F be a field of characteristic $p > 0$. Fix an element c in F. Prove that $f(x) = x^p - c$ is irreducible in $F[x]$ if and only if $f(x)$ has no roots in F.

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6)

(10 pts) a) Determine the Galois closure F of $\mathbb{Q}(\sqrt[3]{1-\sqrt{7}})$. What are possible values of $[F\colon\mathbb{Q}]$?

(15 pts) b) Show that $F/\mathbb{Q}$ is an extension by radicals.

(5 pts) c) Use part b) to conclude that $\mathrm{Gal}(F/\mathbb{Q})$ is solvable.

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7. Let q be a prime and let

$$f_q(x) = x^{q-1} + x^{q-2} + \ldots + 1.$$

(15 pts) a) Suppose a prime number p divides $f_q(a)$ for some integer a. Prove that either $p = q$ or $p \equiv 1 (\text{mod } q)$.

(15 pts) b) Prove there are infinitely many primes of the form $qb + 1$, b an integer.

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8.

(15 pts) a) Show that $\sqrt{3} \notin \mathbb{Q}(\sqrt[3]{2}, \omega)$, where $\omega^2 + \omega + 1 = 0$.

(15 pts) b) Determine the Galois group of

$$f(x) = x^5 - 3x^3 - 2x^2 + 6.$$

(You may assume $\mathrm{Gal}(\mathbb{Q}(\sqrt[3]{2}, \omega)/\mathbb{Q}) \simeq S_3$.)

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