MA 553 **GOINS**

QUALIFYING EXAM – SPRING 2008

This exam is to be done in two hours in one continuous sitting. Begin each question on a new sheet of paper. In answering any part of a question, you may assume the results in previous parts, even if you have not solved them. Be sure to provide *all details of your work*: give definitions of all terms you state, provide references for all theorems you quote, and prove all statements you claim.

Problem 1. Let $(G, \circ)$ be a group, $(H, \star)$ be an abelian group, and $\varphi : G \to H$ be a group homomorphism. If N is a subgroup such that $\ker(\varphi) \leq N \leq G$, show that $N \trianglelefteq G$ is a normal subgroup. [10 points]

Problem 2. Let $(G, \circ)$ be an finite abelian group of even order $|G| = 2k$.

a. For k odd, show that G has exactly one element of order 2. [5 points]
b. Does the same happen for k even? Prove or give a counterexample. [5 points]

Problem 3. Let $(G, \circ)$ be a finite group of odd order, and $H \trianglelefteq G$ be a normal subgroup of prime order $|H| = 17$. Show that $H \leq Z(G)$. [10 points]

Problem 4. Let $(G, \circ)$ be a finite group. Show that there exists a positive integer n such that G is isomorphic to a subgroup of A_n, the alternating group on n letters. (*Hint*: Show that A_n contains a copy of S_{n-2} when $n \geq 3$.) [10 points]

Problem 5. Let $(G, \circ)$ be a group of order $|G| = 200$.

a. Show that G is solvable. [5 points]
b. Show that G is the semidirect product of two p-subgroups. [5 points]

Problem 6. Let $(R, +, \cdot)$ and $(S, +, \cdot)$ be commutative rings with $1 \neq 0$, and let $\varphi : R \to S$ be a surjective ring homomorphism. Assuming that R is *local*, i.e., it has a unique maximal ideal, show that S is also local. [10 points]

Problem 7. Let $(R, +, \cdot)$ be a Principal Ideal Domain.

a. Show that every maximal ideal in R is a prime ideal. [5 points]
b. Must every prime ideal in R be a maximal ideal? Prove or give a counterexample. [5 points]

Problem 8. Let L/F be a Galois extension of degree $[L : F] = 2p$, where p is an odd prime.

a. Show that there exists a unique quadratic subfield E, i.e, $F \subseteq E \subseteq L$ and $[E : F] = 2$. [5 points]
b. Does there exist a unique subfield K of index 2, i.e., $F \subseteq K \subseteq L$ and $[L : K] = 2$? Prove or give a counterexample. [5 points]

Problem 9. Fix a prime p, and consider the *Artin-Schreier* polynomial $f(x) = x^p - x - 1$.

a. Let $\mathbb{F}_p(f)$ be the splitting field of $f(x)$ over $\mathbb{F}_p$. Show that $\mathrm{Gal}\big(\mathbb{F}_p(f)/\mathbb{F}_p\big) \cong Z_p$. [5 points]
b. Prove that $f(x)$ is irreducible in $\mathbb{Z}[x]$. [5 points]

Problem 10. Determine the Galois group of the splitting field over $\mathbb{Q}$ of $f(x) = x^4 + 4$. [10 points]