**MA 553** **GOINS**

**QUALIFYING EXAM – FALL 2006**

This exam is to be done in two hours in one continuous sitting. Begin each question on a new sheet of paper. In answering any part of a question, you may assume the results in previous parts, even if you have not solved them. Be sure to provide *all details of your work*: give definitions of all terms you state, provide references for all theorems you quote, and prove all statements you claim.

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*Problem 1.* Let $(G, \circ)$ be a group. Show that $G$ is abelian whenever $\mathrm{Aut}(G)$ is a cyclic group under composition. [10 points]

*Problem 2.* Let $(G, \circ)$ be an abelian group. The *torsion subgroup of* $G$ is defined as the collection of elements of finite order: $G_{\mathrm{tors}} = \{g \in G \mid g^m = e \text{ for some integer } m > 0\}$.

a. Show that the quotient group $G/G_{\mathrm{tors}}$ is *torsion free* i.e., it contains no nontrivial elements of finite order. [5 points]
b. Show that $G_{\mathrm{tors}}$ is finite whenever $G$ is finitely generated. (Do not assume that $G$ is finite). [5 points]

*Problem 3.* Let $(G, \circ)$ be a group of order $|G| = 351$. Show that $G$ is solvable. [10 points]

*Problem 4.* Let $(G, \circ)$ be a group, and $H \leq G$ be a subgroup of finite index. Show that there exists a normal subgroup $N \trianglelefteq G$ contained in $H$ which is also of finite index. (Do not assume that $G$ is finite.) [10 points]

*Problem 5.* Let $(G, \circ)$ be a finite group, and $\varphi : G \to G$ be a group homomorphism. Show that for all normal Sylow $p$-subgroups $P \trianglelefteq G$ we have $\varphi(P) \leq P$. [10 points]

*Problem 6.* Let $(R, +, \cdot)$ be a commutative ring with $1 \neq 0$.

a. Show that $R$ is an integral domain if and only if $(0)$ is a prime ideal. [5 points]
b. Show that $R$ is a field if and only if $(0)$ is a maximal ideal. [5 points]

*Problem 7.* Let $(R, +, \cdot)$ be a Unique Factorization Domain. Choose an irreducible element $p \in R$, and define the *localization at* $p$ as the ring of fractions $R_p = D^{-1}R$ with respect to the multiplicative set $D = R - (p)$. Show that $R_p$ is a Principal Ideal Domain. [10 points]

*Problem 8.* Let $(F, +, \cdot)$ be a field, and $F(\theta)/F$ be a finite, separable extension. Let $L$ be the splitting field of the minimal polynomial $m_{\theta,F}(x) \in F[x]$. Prove that for every prime $p$ dividing the degree $[L : F]$, there exists a field $K$ such that $F \subseteq K \subseteq L$, $[L : K] = p$, and $L = K(\theta)$. [10 points]

*Problem 9.* Let $(\mathbb{F}_p, +, \cdot)$ be a finite field whose cardinality $p$ is prime. Fix a positive integer $n$ which is not divisible by $p$, and let $\zeta_n$ be a primitive $n$th root of unity. Show that $\left[\mathbb{F}_p(\zeta_n) : \mathbb{F}_p\right] = \alpha$ is the least positive integer such that $p^\alpha \equiv 1 \pmod{n}$. [10 points]

*Problem 10.* Prove that the Galois group of the splitting field over $\mathbb{Q}$ of $f(x) = x^4 + 4\,x^2 + 2$ is a cyclic group. [10 points]