QUALIFYING EXAMINATION
Math 553
August 2005 - Prof. Lipman

BEGIN EACH QUESTION (I-IV) ON A NEW SHEET OF PAPER.

IN ANSWERING ANY PART OF A QUESTION, YOU MAY ASSUME THE RESULTS IN PREVIOUS PARTS, EVEN IF YOU HAVEN'T DONE THEM.

[**Bold numbers**] INDICATE POINTS (**60** TOTAL).

I. Let G be a group of order 24 containing no element of order 6. Let $T < G$ be a Sylow 3-subgroup.
Prove:

(a) [**3**] G has no normal subgroup of order 2.

(b) [**3**] The centralizer of T is T itself.

(c) [**3**] The subgroup T has exactly 4 conjugates.

(d) [**3**] If N_1, N_2, N_3, and N_4 are the normalizers of the conjugates of T, and $i \neq j$, then $N_i \cap N_j$ does not contain a subgroup of order 3.

(e) [**3**] G is isomorphic to the symmetric group S_4.
Hint. Consider the action of G by conjugation on the set of conjugates of T.

II. Let R be a unique factorization domain, let (x_{ij}) $(1 \leq i \leq n,\ 1 \leq j \leq n)$ be a family of independent indeterminates, and let R_{nn} be the polynomial ring $R[x_{11}, x_{12}, \ldots, x_{21}, x_{22}, \ldots, x_{nn}]$. Let $D_n \in R_{nn}$ be the determinant of the $n \times n$ matrix

$$\begin{pmatrix} x_{11} & x_{12} & \ldots & x_{1n} \\ x_{21} & x_{22} & \ldots & x_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ x_{n1} & x_{n2} & \ldots & x_{nn} \end{pmatrix}$$

and let $D_{n-1} \in R_{nn}$ be the cofactor of x_{nn}, i.e., the determinant of the matrix obtained from the above one by deleting the n-th row and the n-th column.
Prove:

(a) [**5**] If $n > 1$ then, in R_{nn}, D_{n-1} does not divide D_n.
Hint. Substitute 1 for x_{n1} and for $x_{i,i+1}$ $(1 \leq i < n)$; and substitute 0 for all other x_{ij}.

(b) [**10**] D_n generates a prime ideal in R_{nn}.
Hint. Expand D_n along the bottom row, and use induction on n.

III. [**15**] Let F be a field and $E = F(c)$ a finite separable field extension of F. Let $K \supset E$ be a splitting field of the minimal polynomial of c over F. Prove that for every prime p dividing the degree $[K : F]$ there exists a field L between F and K such that $[K : L] = p$ and $K = L(c)$.

IV. (a) [**3**] Show that $e^{2\pi i/(6r)}$ $(0 < r \in \mathbb{Z})$ is a root of the polynomial $f(X) := X^{2r} - X^r + 1 \in \mathbb{Q}[X]$.

(b) [**6**] Prove that $f(X)$ in (a) is irreducible if and only if r is of the form $2^a 3^b$ $(a, b \geq 0)$.

(c) [**6**] Prove that $X^{2r} + X^r + 1 \in \mathbb{Q}[X]$ is irreducible if and only if r is a power of 3.