QUALIFYING EXAMINATION
JANUARY 2002
MATH 553 - Prof. Ulrich

1. (13 points) Let G be a group and H a subgroup of finite index. Show that there exists a normal subgroup N of G of finite index with $N \subset H$.

2. (13 points) Show that every group of order 992 $(= 32 \cdot 31)$ is solvable.

3. (16 points) Let G be a group of order 56 with a normal 2-Sylow subgroup Q, and let P be a 7-Sylow subgroup of G. Show that either $G \cong P \times Q$ or $Q \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$. (Hint: P acts on $Q \backslash \{e\}$ via conjugation; show that this action is either trivial or transitive.)

4. (14 points) Let R be a commutative ring and $\mathrm{Rad}(R)$ the intersection of all maximal ideals of R.

(a) Let $a \in R$. Show that $a \in \mathrm{Rad}(R)$ if and only if $1 + ab$ is a unit for every $b \in R$.

(b) Let R be a domain and $R[X]$ the polynomial ring over R. Deduce that $\mathrm{Rad}(R[X]) = 0$.

5. (14 points) Let R be a unique factorization domain and P a prime ideal of $R[X]$ with $P \cap R = 0$.

(a) Let n be the smallest possible degree of a non-zero polynomial in P. Show that P contains a primitive polynomial f of degree n.

(b) Show that P is the principal ideal generated by f.

6. (16 points) Let k be a field of characteristic zero. Assume that every polynomial in $k[X]$ of odd degree and every polynomial in $k[X]$ of degree two has a root in k. Show that k is algebraically closed.

7. (14 points) Let $k \subset K$ be a finite Galois extension with Galois group $G(K/k)$, let L be a field with $k \subset L \subset K$, and set $H = \{\sigma \in G(K/k) \mid \sigma(L) = L\}$.

(a) Show that H is the normalizer of $G(K/L)$ in $G(K/k)$.

(b) Describe the group $H/G(K/L)$ as an automorphism group.