Heinzer Math 553 Qualifying Exam 12 August 2002

Let $\mathbb{Z}$ denote the ring of integers and $\mathbb{Q}, \mathbb{C}$ the fields of rational and complex numbers, respectively.

(20) 1. Let $\mathbb{Q}(x)$ denote the field of rational functions in an indeterminate x with coefficients from $\mathbb{Q}$.

(i) Describe the group $\mathrm{Aut}(\mathbb{Q}(x)/\mathbb{Q}(x^2))$ of automorphisms of $\mathbb{Q}(x)$ that fix the field $\mathbb{Q}(x^2)$. Is the field extension $\mathbb{Q}(x)/\mathbb{Q}(x^2)$ Galois ?

(ii) Describe the group $\mathrm{Aut}(\mathbb{Q}(x)/\mathbb{Q}(x^3))$ of automorphisms of $\mathbb{Q}(x)$ that fix the field $\mathbb{Q}(x^3)$. Is the field extension $\mathbb{Q}(x)/\mathbb{Q}(x^3)$ Galois ?

(iii) Let $f = x^2 - x$. Describe the group $\mathrm{Aut}(\mathbb{Q}(x)/\mathbb{Q}(f))$ of automorphism of $\mathbb{Q}(x)$ that fix the field $\mathbb{Q}(f)$. Is the field extension $\mathbb{Q}(x)/\mathbb{Q}(f)$ Galois ?

(iv) Let $\phi : \mathbb{Q}(x) \to \mathbb{Q}(x)$ be the automorphism determined by defining $\phi(x) = x + 1$ and let $K = \{r \in \mathbb{Q}(x) \quad | \quad \phi(r) = r\}$ be the fixed field of ϕ. What is $[\mathbb{Q}(x) : K]$? What is $[K : \mathbb{Q}]$?

(v) For $f = x^2 - x$, what is the field $\mathbb{Q}(x^2) \cap \mathbb{Q}(f)$?

Recall that if R and S are rings, then $R \times S = \{(r,s) \mid r \in R, s \in S\}$ is a ring where addition and multiplication in $R \times S$ are defined componentwise.

(6) 2. Describe all the prime ideals of $\mathbb{Z} \times \mathbb{Z}$.

(16) 3. Consider the polynomial ring $\mathbb{Z}[x]$.

(i) Define $\phi_1 : \mathbb{Z}[x] \to \mathbb{Z}$, by $\phi_1(f(x)) = f(1)$ for every $f(x) \in \mathbb{Z}[x]$. Give generator(s) for the ideal $\ker \phi_1$.

(ii) Define $\phi_{-1} : \mathbb{Z}[x] \to \mathbb{Z}$, by $\phi_{-1}(f(x)) = f(-1)$ for every $f(x) \in \mathbb{Z}[x]$. Give generator(s) for the ideal $\ker \phi_{-1}$.

(iii) Define $\phi : \mathbb{Z}[x] \to \mathbb{Z} \times \mathbb{Z}$ by $\phi(f(x)) = (f(1), f(-1))$ for every $f(x) \in \mathbb{Z}[x]$. Give generator(s) for the ideal $\ker \phi$.

(iv) Prove or disprove that ϕ is surjective.

(20) 4. Let K/F be a finite separable algebraic field extension and let $\alpha \in K$.

(i) Define the norm $N_{K/F}(\alpha)$ of α from K to F.

(ii) Prove that $N_{K/F}(\alpha) \in F$.

(iii) Define the trace $Tr_{K/F}(\alpha)$ of α from K to F.

(iv) Prove that $Tr_{K/F}(\alpha) \in F$.

(v) For $K = \mathbb{Q}(\sqrt[3]{2})$, compute $N_{K/\mathbb{Q}}(2\sqrt[3]{2})$ and $Tr_{K/\mathbb{Q}}(2\sqrt[3]{2})$.

(vi) For $K = \mathbb{Q}(\sqrt[3]{2})$, compute $N_{K/\mathbb{Q}}(2)$ and $Tr_{K/\mathbb{Q}}(2)$.

(8) 5. Find all subgroups of the cyclic group $Z_{45} = \langle x \rangle$, giving a generator for each. Diagram the lattice of subgroups.

(8) 6. Diagram the lattice of ideals of the ring $R = \mathbb{Z}[x]/(15, x^2 + 1)$. What is the cardinality of R?

(6) 7. Suppose $\alpha \in \mathbb{C}$ is algebraic over $\mathbb{Q}$.

(i) Define "*α can be expressed by radicals*" or the equivalent phrase "*α can be solved for in terms of radicals.*"

(ii) For a polynomial $f(x) \in \mathbb{Q}[x]$, define "*$f(x)$ can be solved by radicals.*"

(14) 8. For n a positive integer, let Z_n denote a cyclic group of order n.

(i) What is the order of the group $\operatorname{Aut}(Z_n)$ of automorphism of Z_n? Explain your answer.

(ii) Are the groups $\operatorname{Aut}(Z_7)$ and $\operatorname{Aut}(Z_9)$ isomorphic? Justify your answer.

(iii) Are the groups $\operatorname{Aut}(Z_8)$ and $\operatorname{Aut}(Z_{12})$ isomorphic? Justify your answer.

(16) 9. Let $\omega \in \mathbb{C}$ be a primitive 9-th root of unity.

(i) What is $[\mathbb{Q}(\omega) : \mathbb{Q}]$?

(ii) List the distinct conjugates of $\omega + \omega^{-1}$ over $\mathbb{Q}$.

(iii) What is the group $\mathrm{Aut}(\mathbb{Q}(\omega + \omega^{-1})/\mathbb{Q})$? Is $\mathbb{Q}(\omega + \omega^{-1})$ Galois over $\mathbb{Q}$?

(iv) Diagram the lattice of subfields of $\mathbb{Q}(\omega)$ giving generators for each.

(12) 10. Let F be a field. For each nonconstant monic polynomial $f = f(x) \in F[x]$, let x_f be an indeterminate. Consider the polynomial ring $R = F[\{x_f\}]$, and let I be the ideal of R generated by the polynomials $f(x_f)$, where f varies over all the nonconstant monic polynomials in $F[x]$.

(i) Prove that $I \neq R$.

(ii) Prove that there exists an extension field K of F in which each nonconstant monic polynomial $f \in F[x]$ has a root.

(8) 11. Let K/F be an algebraic field extension. Suppose R is a subring of K such that $F \subseteq R$. Prove or disprove that R is a field.

(18) 12. (i) Let $K/\mathbb{Q}$ be the splitting field of the polynomial $x^5 - 1 \in \mathbb{Q}[x]$. Diagram the lattice of subfields of $K/\mathbb{Q}$. For each subfield, give generators and list its degree over $\mathbb{Q}$.

(ii) Let $L/\mathbb{Q}$ be the splitting field of the polynomial $x^5 - 2 \in \mathbb{Q}[x]$. Diagram the lattice of subfields of $L/\mathbb{Q}$. For each subfield, give generators and list its degree over $\mathbb{Q}$.

(8) 13. Diagram the lattice of subgroups of the dihedral group D_8.

(10) 14. Let $G = \{z \in \mathbb{C} \mid z^n = 1 \text{ for some } n \in \mathbb{Z}^+\}$. Define $\phi : G \to G$ by $\phi(z) = z^4$.

(i) What is the order of $\ker(\phi)$?

(ii) Prove or disprove that ϕ is surjective.

(8) 15. Let $\mathbb{F}_3$ denote the field with 3 elements. Prove or disprove that the polynomial ring $\mathbb{F}_3[x]$ has infinitely many nonassociate prime elements.

(6) 16. Let R be a commutative ring with 1.

(i) Define the *characteristic* of R.

(ii) Does there exist a ring having characteristic 6? Justify your answer.

(8) 17. (i) Does there exist a field having 6 elements? If so, describe how to obtain such a field; if not, explain why not.

(ii) Does there exist a field having 25 elements? If so, describe how to obtain such a field; if not, explain why not.

(8) 18. Suppose H and K are normal subgroups of a group G and that $H \cap K = 1$, where 1 denotes the identity subgroup. If $x \in H$ and $y \in K$ is it always true that $xy = yx$? Justify your answer.