Math 553 **Qualifying Examination** **January, 1997**

Each problem 1–4 is worth 10 points, and #5 is worth 20. In working any part of a problem you may assume the preceding parts, even if you haven't done them.

1. Let G be a group of finite order n, and let d be an integer relatively prime to n.

(a) Show that there is an integer a such that every $x \in G$ satisfies $x^{ad} = x$.

(b) Show that for every $y \in G$, there is precisely one $x \in G$ such that $x^d = y$.

2. Let p be a prime dividing the order of the finite group G, and let P be a Sylow p-subgroup of G. Let a and $b = zaz^{-1}$ ($z \in G$) both lie in $Z(P)$, the centralizer of P. Show that $z^{-1}Pz \subset Z(a)$; and deduce that $b = yay^{-1}$ for some y in the normalizer of P.

3. Let i be an element of a commutative ring S such that i is *idempotent,* i.e., $i^2 = i$.

(a) Prove that the principal ideal iS is a ring, with identity element i.

(b) Let $i' = 1 - i$. Show that i' is idempotent, and establish a ring-isomorphism

$$S \xrightarrow{\sim} (iS) \times (i'S).$$

4. (a) Factor 2 into primes in $\mathbb{Z}[\sqrt{-1}]$ and in $\mathbb{Z}[\sqrt{-2}]$. (Justify your answer.)

[An element a in a ring R is defined to be *prime* if the ideal aR is prime.]

(b) Show for any integer $n > 2$ that in $\mathbb{Z}[\sqrt{-n}]$, 2 is irreducible but not prime.

(c) For which positive integers n is $\mathbb{Z}[\sqrt{-n}]$ a unique factorization domain? (Answer only—no justification required.)

5. Let $L = \mathbb{Q}(\sqrt{2}, \sqrt{3})$ where $\mathbb{Q}$ is the field of rational numbers. You may assume that $[L : \mathbb{Q}] = 4$.

(a) For rational numbers a, b, c, d with at least two of b, c, d non-zero, prove that

$$L = \mathbb{Q}(a + b\sqrt{2} + c\sqrt{3} + d\sqrt{6}).$$

(b) Let $\alpha = -(2 + \sqrt{2})(2 + \sqrt{3})(3 + \sqrt{6})$. Show that any $\mathbb{Q}$-conjugate of α is of the form $\alpha\ell^2$ with $\ell \in L$; and deduce that $K := L(\sqrt{\alpha})$ is a degree 8 galois extension of $\mathbb{Q}$.

(c) Show that L is the fixed field of any non-identity automorphism θ of K such that $\theta^2 =$ identity. Hint: Consider $(\sqrt{\alpha})(\theta\sqrt{\alpha})$.

(d) Is the galois group of $K/\mathbb{Q}$ abelian, dihedral, or quaternionic? (Justify your answer.)