QUALIFYING EXAMINATION
MATH 553

August 1997

Write down the answer to the following questions with your reasoning. If your reasoning is correct, even when your final answer happens to be wrong, will get a substantial amount of credit, while providing a final answer without any reasoning may not get full credit.

1. Let $G = \mathbb{Z}/(30)$ be the cyclic group of order 30.

(i) (10 points) What is the total number of subgroups of G (including G itself and the trivial group consisting only of the identity) ?

(ii) (10 points) What is the total number of endomorphisms (group homomorphisms from G to G itself) ?

(iii) (20 points) Prove that

$$Aut(G) \cong \mathbb{Z}/(4) \times \mathbb{Z}/(2)$$

where $Aut(G)$ is the group of automorphisms of G.

2. (40 points) Prove that there are only two non-isomorphic groups of order $2p$, where p is an odd prime, and describe these two groups.

3. Let G be a finite group.

(i) (10 points) For an element $g \in G$, define the centralizer $C(g)$ of g to be

$$C(g) = \{h \in G; hg = gh\}.$$

Show that $|C(g)| = |C(g')|$ if g and g' are conjugate to each other, i.e., if $g' = f^{-1}gf$ for some $f \in G$. $|H|$ denotes the order of a group H.

(ii) (20 points) Let $g_1, \cdots, g_l$ be a complete set of representatives from the conjugacy classes of G. (The number l of conjugacy classes is called the class number.) Prove that

$$\frac{1}{|C(g_1)|} + \frac{1}{|C(g_2)|} + \cdots + \frac{1}{|C(g_l)|} = 1.$$

(iii) (10 points) Find all the triplets of positive integers (c_1, c_2, c_3) such that

$$c_1 \leq c_2 \leq c_3$$

and that

$$\frac{1}{c_1} + \frac{1}{c_2} + \frac{1}{c_3} = 1.$$

(iv) (20 points) Find all the finite groups with the class number $l = 3$.

4. Let

$$R = \{\alpha + \beta\omega; \alpha, \beta \in \mathbb{Z}\} \text{ where } \omega = \frac{-1 + \sqrt{-3}}{2}.$$

(i) (20 points) Find all the units in the ring R.
(ii) (20 points) Show that R is a unique factorization domain.

5. Let $L = \mathbb{Q}[\sqrt[3]{5}, \omega]$ be an extension of the field of rational numbers $\mathbb{Q}$, where $\omega = \frac{-1+\sqrt{-3}}{2}$.

(i) (20 points) Show that the Galois group $G(L/\mathbb{Q})$ is isomorphic to S_3, the symmetric group of degree 3.

(ii) (20 points) Let $H \subset G$ be the unique Sylow 3-subgroup of G. Find an element $\alpha \in L$ such that $L^H = \mathbb{Q}[\alpha]$ where L^H is the intermediate field $\mathbb{Q} \subset L^H \subset L$ fixed by H.

6. Let $f(x) = x^4 + 4x^2 + 2$.
(i) (20 points) Prove that $f(x)$ is irreducible over $\mathbb{Q}$.
(ii) (20 points) Let L be the splitting field of $f(x)$ over $\mathbb{Q}$. Determine the Galois group $G(L/\mathbb{Q})$.

7. (30 points) Let $\mathbb{F}_q$ be a finite field with q elements and $f(x) \in \mathbb{F}_q[x]$ be an irreducible polynomial. Show that $f(x)$ divides $x^{q^n} - x$ if and only if the degree of $f(x)$ divides n.