Qualifying Examination

JANUARY, 1996

Math 553

In answering any part of a problem you may assume the answers to the preceding parts.

The number of [points] carried by a correct answer is indicated after each question.

1. Prove that each group of order 616 is solvable. [10]

2. Let G be a group, which contains two elements x and y which commute and have finite orders m and n, respectively. Prove that G contains an element z of order equal to the least common multiple of m and n. [10]

3. Let p be a prime number, let $\mathbb{F}_p$ be the field with p elements, let S_p be the ring of all 2×2 matrices with elements in $\mathbb{F}_p$, and set:

$$F_p = \{ \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \in S_p \mid a \in \mathbb{F}_p \} ; \qquad R_p = \{ \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \in S_p \mid a, b \in \mathbb{F}_p \} .$$

(a) Prove that $F_p \cong \mathbb{F}_p$ and that R_p is a commutative subring of S_p. [5]

(b) Prove that $R_p = F_p[x]$ for some $x \in S_p$. [5]

(c) Prove that R_p has non-zero nilpotent elements if and only if $p = 2$. [10]

(d) Prove that R_p is a field if and only if $p = 4k - 1$ for some integer $k \geq 1$. [10]

(e) Describe the group of units U_p of R_p as a direct product of cyclic groups. [5]

(f) Write down a matrix A_p of highest order in U_p for $p = 2, 3, 5$. [5]

4. Let R be a UFD with field of fractions F, and let f and g be polynomials in $R[x]$ which have no common root in any field extension of F.

(a) Prove that there exist polynomials $h, k \in R[x]$ and a non-zero element $d \in R$ such that $fh + gk = d$. Is d the greatest common divisor of f and g? Justify your answer. [5]

(b) When R is the ring of integers $\mathbb{Z}$, prove that the ideal I generated by $f = x^2 + 4x + 5$ and $g = x^2 + x + 1$ is maximal, and determine the field $F = \mathbb{Z}[x]/I$. [10]

5. Let F be a (not necessarily finite) field extension of the field of rational numbers $\mathbb{Q}$, and let $\sigma \colon F \to F$ be a ring homomorphism.

(a) Prove that if F is algebraic over $\mathbb{Q}$, then σ is an isomorphism. [10]

(b) Show that the conclusion may fail if F is not assumed algebraic over $\mathbb{Q}$. [5]

6. Let $\mathbb{Q} \subset F$ be a Galois extension with Galois group the symmetric group S_4. List all the numbers that occur as degrees of minimal polynomials over $\mathbb{Q}$ of elements $x \in F$, and justify your answer. [10]