(16) 1. Let $\mathbb{F}_p$ denote the prime field with p elements.

(i) How many monic polynomials are there in $\mathbb{F}_p[x]$ of degree 2?

(ii) How many monic irreducible polynomials are there in $\mathbb{F}_p[x]$ of degree 2?

(iii) How many monic polynomials are there in $\mathbb{F}_p[x]$ of degree 3 that have a multiple root?

(iv) How many monic irreducible polynomials are there in $\mathbb{F}_p[x]$ of degree 3?

(12) 2. Let F be a field. Prove that in the polynomial ring $F[x]$ there are infinitely many irreducible polynomials.

(10) 3. Suppose α is a complex number that is algebraic over $\mathbb{Q}$ and that $[\mathbb{Q}(\alpha) : \mathbb{Q}]$ is odd. Prove that $\mathbb{Q}(\alpha) = \mathbb{Q}(\alpha^2)$.

(10) 4. For a prime number p and a nonzero $a \in \mathbb{F}_p$, where $\mathbb{F}_p$ is the field with p elements, prove that the polynomial $x^p - x + a$ is irreducible and separable over $\mathbb{F}_p$.

(10) 5. Let $\omega \in \mathbb{C}$ be a primitive 10-th root of unity.

(i) What is $[\mathbb{Q}(\omega) : \mathbb{Q}]$?

(ii) Diagram the lattice of subfields of $\mathbb{Q}(\omega)$ giving generators for each.

(20) 6. Let G be a finite group and let p be a prime number dividing $|G|$. Let $\mathcal{S}$ denote the set of p-tuples of elements of G the product of whose coordinates is 1: thus $\mathcal{S} = \{(x_1, x_2, \ldots, x_p) \mid x_i \in G \quad \text{and} \quad x_1 x_2 \cdots x_p = 1 \}$.

(i) What is the cardinality of $\mathcal{S}$?

(ii) For $\alpha, \beta \in \mathcal{S}$, define $\alpha \sim \beta$ if β is a cyclic permutation of α. What is needed for $\sim$ to be an equivalence relation on $\mathcal{S}$?

(iii) Assuming $\sim$ is an equivalence relation, which equivalence classes with respect to $\sim$ contain exactly one element?

(iv) What integers are the order of an equivalence class with respect to $\sim$? Justify your answer.

(v) Prove that G has an element of order p.

(20) 7. Let F be an infinite field and let K/F be a finite algebraic field extension.

(i) If $K = F(\alpha)$ for some $\alpha \in K$, prove that there are only finitely many subfields of K that contain F.

(ii) If there are only finitely many subfields of K that contain F, prove that $K = F(\alpha)$ for some $\alpha \in K$.

(15) 8. Let $\mathbb{Z}$ denote the ring of integers and let x be an indeterminate over $\mathbb{Z}$.

(i) Is every ideal of $\mathbb{Z}[x]/(x^2 - 1)$ principal? Justify your answer.

(ii) Is every ideal of $\mathbb{Z}[x]/(x^2)$ principal? Justify your answer.

(iii) Is every ideal of $\mathbb{Z}[x]/(15)$ principal? Justify your answer.

(12) 9. Let $\mathbb{Z}$ denote the ring of integers and let x be an indeterminate over $\mathbb{Z}$. Diagram the lattice of ideals of the ring $\mathbb{Z}[x]/(6, x^3)$.

(10) 10. Suppose $f(x) \in \mathbb{Q}[x]$ is a monic polynomial of degree 5 that is reducible in $\mathbb{Q}[x]$ and let $L/\mathbb{Q}$ be a splitting field of $f(x)$. List all positive integers n that are possibly equal to $[L : \mathbb{Q}]$.

(12) 11. Suppose L/K is a separable normal field extension with $[L : K] = 21$.

(i) What integers n are possibly equal to the degree of a monic irreducible polynomial $f(x) \in K[x]$ for which L/K is a splitting field of $f(x)$?

(ii) If the Galois group of L/K is known to be an abelian group, what integers n are possibly equal to the degree of a monic irreducible polynomial $f(x) \in K[x]$ for which L/K is a splitting field of $f(x)$? Justify your answer.

(16) 12. Let G be a finite group with $|G| = n$.

(i) Prove that G is isomorphic to a subgroup of the symmetric group S_n.

(ii) Is G isomorphic to a subgroup of the alternating group A_m for some positive integer m? Justify your answer.

(15) 13. Suppose G_i is a group and H_i is a normal subgroup of G_i, $i = 1, 2$. Let "$\cong$" denote "is group isomorphic to".

(i) If $H_1 \cong H_2$ and $G/H_1 \cong G/H_2$, does it follow that $G_1 \cong G_2$? Justify your answer.

(ii) If $G_1 \cong G_2$ and $H_1 \cong H_2$, does it follow that $G_1/H_1 \cong G_2/H_2$? Justify your answer.

(iii) If $G_1 \cong G_2$ and $G_1/H_1 \cong G_2/H_2$, does it follow that $H_1 \cong H_2$? Justify your answer.

(12) 14. Let R be an integral domain and let $r \in R$ be a nonzero nonunit.

(i) What does it mean for r to be irreducible?

(ii) What does it mean for r to be prime?

(iii) Prove that if r is prime, then r is irreducible.

(10) 15. Suppose H is a subgroup of a group G such that $[G : H] = 6$. Prove there exists a normal subgroup N of G such that $N \subseteq H$ and $[G : N] \leq 6!$.