Qualifying Examination

August, 1995

Math 553

A correct answer to each part of each problem below is worth 10 points*. In answering any part of a question you may assume the preceding parts.

Notation: $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$, and $\mathbb{F}_p$ denote, respectively, the fields of rational numbers, real numbers, complex numbers, and with p elements.

1. Let p and q be distinct prime numbers. Prove that every group of order p^2q is solvable.

2. Let S_4 be the symmetric group on 4 elements, and let B be the group of permutations of the set $S = \{P_1, \ldots, P_s\}$ of Sylow 3-subgroups of S_4.

(a) List the subgroups $P_1, \ldots, P_s$ and justify your answer.

(b) Prove that the map $S_4 \to B$, which sends each $g \in S_4$ to the permutation $\begin{pmatrix} P_1 & P_2 & \cdots & P_s \\ g(P_1)g^{-1} & g(P_2)g^{-1} & \cdots & g(P_s)g^{-1} \end{pmatrix} \in B$, is an isomorphism of groups.

3. Determine which polynomials are irreducible in the given ring:

(a) $f(x) = x^4 + x^3 + 3x^2 + 2 \in \mathbb{Q}[x]$.

(b) $f(x, y) = 1 + x + y^2 + xy^2 + yx^2 \in \mathbb{Q}[x, y]$.

4. Prove the following assertions for the ring $S = \mathbb{R}[x]/\big((x^2+1)^2\big)$.

(a) There exist exactly two ring homomorphisms $\pi\colon S \to \mathbb{C}$ such that $\pi|_{\mathbb{R}} = \mathrm{id}_{\mathbb{R}}$.

(b) Choose one of the homomorphisms in (a), and call it π. Prove that there exists a unique homomorphism of rings $\sigma\colon \mathbb{C} \to S$ such that $\sigma|_{\mathbb{R}} = \mathrm{id}_{\mathbb{R}}$ and $\pi\sigma = \mathrm{id}_{\mathbb{C}}$.

[Hint: Prove and use the following: if such a σ exists, then $(\sigma(i))^2 = -1$ and $\pi\sigma(i) = \pi(x)$.]

(c) The homomorphism σ extends to an isomorphism of rings $\mathbb{R}[y]/(y^2) \to S$.

5. Let p be a prime number, let L be the field of rational functions $\mathbb{F}_p(x, y)$, and let F be the subfield $\mathbb{F}_p(x^p, y^p) \subseteq L$. For each integer $n \geq 1$ consider the element $z_n = x + yx^{p^n} \in L$ and the subfield $E_n = F(z_n) \subseteq L$. Prove the following assertions.

(a) $(L : F) = p^2$ and $(E_n : F) = p$.

(b) $E_n \neq E_m$ if $n \neq m$.

6. Let $K = \mathbb{Q}(\sqrt{5}, \sqrt{6}, \sqrt{7})$. Prove the following assertions.

(a) $(K : \mathbb{Q}) = 8$.

* Only the first 100 points you collect will count towards your score.

(b) K is a Galois extension of $\mathbb{Q}$ and $\mathrm{Gal}(K|\mathbb{Q}) \cong C_2 \times C_2 \times C_2$, where C_2 is the cyclic group of order 2.