Qualifying Exam for Math 553
August 1994

1) (10 points) Let $\mathbb{Z}$ be the group of integers. Determine the number of group homomorphisms from $\mathbb{Z}/m\mathbb{Z}$ to $\mathbb{Z}/n\mathbb{Z}$ for any two positive integers n and m.

2) (10 points) Let p be a prime number. Let $\mathbf{GL}(2,\ \mathbb{Z}_p)$ be the group of all 2×2 invertible matrices over $\mathbb{Z}/p\mathbb{Z}$. Find the order of $\mathbf{GL}(2,\ \mathbb{Z}_p)$.

3) (10 points) Describe all Sylow p-subgroups of $\mathbb{S}_5$ and an element of maximal order in $\mathbb{S}_5$.

4) (10 points) Prove or disprove: The group of rigid motions, i.e., all mappings which save distance, of the real plane $\mathbb{R}^2$ is generated by all reflections.
[Hint: Any rigid motion is determined by the images of any three points.]

5) (10 points) Is the polynomial ring $\mathbb{Z}[x]$ over the ring of integers a Euclidean domain? P.I.D? U.F.D.? Prove your assertions.

6) (10 points) Let m, n, q, s be four non-negative integers. Show that

$$(x^3 + x^2 + x + 1) \mid (x^{4m+3} + x^{4n+2} + x^{4q+1} + x^{4s})$$

in the polynomial ring $\mathbb{Z}[x]$ over integers.

7) (10 points) Let $\mathbf{S}$ be a ring and $f(x)$ be a zero-divisor in $\mathbf{S}[x]$. Prove that there is an element $a \neq 0$ in $\mathbf{S}$ such that $af(x) = 0$.

8) (10 points) Let $\mathbb{Q}$ be the field of all rationals. Find the minimal monic polynomial $f(x)$ of $\sqrt{3} + \sqrt[3]{3}$ over $\mathbb{Q}$, and the Galois group of the splitting field of $f(x)$ over $\mathbb{Q}$.

9) Let $\mathbf{K}$ be a field and x, y variables. Let a, b, c, d be integers with

$$n = \left| \, det \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right| \quad = \mid ad - bc \mid \neq 0$$

Let $\mathbf{L}$=$\mathbf{K}(x, y)$, and $\mathbf{S}$=$\mathbf{K}(x^a y^b, x^c y^d)$.

(a) (10 points) Show that $\mathbf{L}$ is a finite extension of $\mathbf{S}$ and $[\mathbf{L}:\mathbf{S}]=n$.

(b) (10 points) Suppose that $\mathbf{K}=\mathbb{C}$ the field of complex numbers. Show that $\mathbf{L}$ is a Galois extension of $\mathbf{S}$ and find the Galois group.
[Hint: $\mathbf{K}(x, y)$=$\mathbf{K}(x, x^s y)$, and $\mathbf{K}(x^a y^b, x^c y^d)$ =$\mathbf{K}(x^a y^b, x^{c+sa} y^{d+sb})$ for any integer s.]