Analysis Qualifying Exam

Last Name: ____________________

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Instructions: Do not open this exam until instructed to do so. Please print your name and student ID number above, and circle the number of your discussion section. Your signature above indicates that you agree to the exam guidelines that are available on our Brightspace page. In particular, you agree to not collaborate with others and to not use outside sources, for example Chegg. Please make sure **your phone is silenced** and stowed where you cannot see it. You may use any available space on the exam for scratch work. If you need more scratch paper, please ask one of the proctors. You must **show your work** (whenever relevant) to receive credit. Please label your scratch work and finalized proofs, if needed.

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1. (a) Show that if $f : \mathbb{R} \to \mathbb{R}$ is continuous almost everywhere, then f is measurable.

(b) Describe a sequence of functions $f_n : [0,1] \to \mathbb{R}$ so that

$$\lim_{n\to\infty} \int_0^1 f_n \, d\lambda = 0$$

but f_n does not converge for any $x \in [0,1]$.

(c) Show that for your example, we can find a subsequence $\{f_{n_k}\}$ so that

$$\lim_{k \to \infty} \int_0^1 |f_{n_k}| \, d\lambda = 0.$$

2. Let $E \subseteq [0,2]$ be a measurable subset and define $f(x) = \lambda(E \cap (-\infty, x])$. Show that f is absolutely continuous. Compute f' and $\int_{-1}^{1} f' \, d\lambda$.

3. Let $f_n \in L^1(X, \mathcal{S}, \mu)$ be a Cauchy sequence. Show that for all $\varepsilon > 0$, there exists $\delta > 0$ so that for all $n \in \mathbb{N}$, we have that if $E \in \mathcal{S}$ satisfies $\mu(E) < \delta$, then

$$\int_E |f_n| \, d\mu < \varepsilon.$$

Hint: you may use that if $f \in \mathcal{L}^1$, then for all $\varepsilon > 0$, there exists $\delta > 0$ so that if $\mu(E) < \delta$, then $\int_E |f| \, d\mu < \varepsilon$.

4. Let $(X, \mathcal{S}, \mu)$ be a finite measure space, $1 \leq p_1 < p_2 \leq \infty$ and let $T : L^{p_2} \to L^{p_1}$ be the inclusion map. Show that T is a bounded linear map. What is $\|T\|$?

5. Let $p \in [1, \infty)$. A sequence $\{a(n)\}$ is called *finitely supported* if there exists some N so that $a(n) = 0$ for all $n > N$.

   (a) Show that if $a = \{a(n)\}$ is finitely supported, then $a \in \ell^p$.

   (b) Show that finitely supported sequences are dense in ℓ^p.

(c) Show that all finitely supported sequences are elements of ℓ^∞, but finitely supported sequences are NOT dense in ℓ^∞.

6. Let $(X, \mathcal{S}, \mu)$ be a σ-finite measure space, $f : X \to \mathbb{R}$ be a non-negative measurable function, and

$$G_f := \{(x, y) : y \leq f(x)\} \subseteq X \times [0, \infty]$$

(a) Show $\int_X f \, d\mu = (\mu \times \lambda)(G_f)$.

(b) Show $\int_X f\, d\mu = \int_0^\infty \mu\{x \in X : y \leq f(x)\}\, d\lambda(y)$