QUALIFYING EXAM
REAL ANALYSIS
WINTER 2022

Observation. You have two hours to complete this exam. Books, notebooks, and any other course materials are NOT allowed. Cell phones must be turned off. No computers or calculators are accepted. Each problem should be solved on a distinct/new page (if you need more space ask for supplementary paper).

Problem	Points
1	
2	
3	
4	
5	
Total	

ex.1 (30p) Label the following statements as true or false in the table provided at the end of the exercise (**T** for true and **F** for false). **If the answer to a statement is false then you are required to provide a counterexample and explain it. No justification is required if the answer is true.**

(1) If $\{f_n\}_n \subset L^1(\mathbb{R})$ with $f_n \overset{n\to\infty}{\longrightarrow} f$ in $\|\cdot\|_{L^1(\mathbb{R})}$ norm then $f_n \overset{n\to\infty}{\longrightarrow} f$ almost everywhere.

(2) If $\{f_n\}_n \subset L^1(\mathbb{R})$ with $f_n \overset{n\to\infty}{\longrightarrow} f$ almost everywhere and $|f_n(x)| \leq 1$ a.e. $x \in \mathbb{R}$ for all $n \in \mathbb{N}$ then $f_n \overset{n\to\infty}{\longrightarrow} f$ in $\|\cdot\|_{L^1(\mathbb{R})}$ norm.

(3) If $\{f_n\}_n \subset L^1(\mathbb{R})$ with $f_n \overset{n\to\infty}{\longrightarrow} f$ in $\|\cdot\|_{L^1(\mathbb{R})}$ norm then $f_n \overset{n\to\infty}{\longrightarrow} f$ in measure.

(4) If $\{f_n\}_n \subset L^1(\mathbb{R})$ with $f_n \overset{n\to\infty}{\longrightarrow} f$ in measure then there exists a subsequence $\{f_{n_k}\}_k$ such that $f_{n_k} \overset{k\to\infty}{\longrightarrow} f$ in $\|\cdot\|_{L^1(\mathbb{R})}$ norm.

(5) If $\{f_n\}_n \subset L^1(\mathbb{R})$ with $f_n \overset{n\to\infty}{\longrightarrow} f$ in $\|\cdot\|_{L^1(\mathbb{R})}$ norm then there exists a subsequence $\{f_{n_k}\}_k$ such that $f_{n_k} \overset{k\to\infty}{\longrightarrow} f$ almost everywhere.

(6) If $\{f_n\}_n \subset L^1(\mathbb{R})$ with $f_n \overset{n\to\infty}{\longrightarrow} f$ almost everywhere then there exists a subsequence $\{f_{n_k}\}_k$ such that $f_{n_k} \overset{k\to\infty}{\longrightarrow} f$ in $\|\cdot\|_{L^1(\mathbb{R})}$ norm.

(7) If $\{f_n\}_n \subset L^1(\mathbb{R})$ with $f_n \overset{n\to\infty}{\longrightarrow} f$ in measure then there exists a subsequence $\{f_{n_k}\}_k$ such that $f_{n_k} \overset{k\to\infty}{\longrightarrow} f$ almost everywhere.

(8) If $\{f_n\}_n \subset L^2(\mathbb{R})$ with $f_n \overset{n\to\infty}{\longrightarrow} f$ in $\|\cdot\|_{L^2(\mathbb{R})}$ norm then $\int_{\mathbb{R}} f_n\, g \overset{n\to\infty}{\longrightarrow} \int_{\mathbb{R}} f\, g$ for any $g \in L^2(\mathbb{R})$.

(9) If $\{f_n\}_n \subset L^2(\mathbb{R})$ with $f_n \overset{n\to\infty}{\longrightarrow} f$ in measure then $\int_{\mathbb{R}} f_n\, g \overset{n\to\infty}{\longrightarrow} \int_{\mathbb{R}} f\, g$ for any $g \in L^2(\mathbb{R})$.

	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)
Answer									

Solution Ex.1 (continuation)

ex.2 (30p) i) Let E be a subset[1] of $[0,1]$ with positive exterior (Lebesgue) measure, that is, $m_*(E) > 0$. Prove that *for any* $0 < \alpha < 1$ *there exists* an open interval $I \subseteq [0,1]$ such that

$$m_*(E \cap I) \geq \alpha\, m_*(I)\,.$$

ii) Prove that if $E \subseteq [0,1]$ measurable such that *there exists* $\alpha > 0$ with

$$m(E \cap I) \geq \alpha\, m(I)$$

for any $I \subseteq [0,1]$ interval then $m(E) = 1$.

Justify in detail all the steps in your proof.

[1] We are not requiring at this stage that E is Lebesgue measurable.

Solution Ex.2 (continuation)

ex.3 (25p) Suppose $f \in L^1(\mathbb{R})$ satisfies

$$\limsup_{\epsilon \to 0} \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{|f(x)|\,|f(y)|}{|x-y|^2 + \epsilon^2}\, dx\, dy < \infty\,.$$

Show that $f(x) = 0$ almost everywhere $x \in \mathbb{R}$.

<u>Present all your reasonings in great detail.</u>

Solution Ex.3 (continuation)

ex.4 (20p) i) Show that if $p,\, q,\, r \geq 1$ such that $\frac{1}{p}+\frac{1}{q}+\frac{1}{r} = 1$ and $f \in L^p(\mathbb{R})$, $g \in L^q(\mathbb{R})$ and $h \in L^r(\mathbb{R})$ then $f\, g\, h \in L^1(\mathbb{R})$.

ii) Let $f \in L^2_{loc}(\mathbb{R})$ and $g \in L^3_{loc}(\mathbb{R})$. Assume that for any positive integer $n \geq 1$ we have

$$\int_{n\leq|x|\leq 2n} |f(x)|^2\, dx \leq 1 \qquad \text{and} \qquad \int_{n\leq|x|\leq 2n} |g(x)|^3\, dx \leq \frac{1}{n}\,.$$

Prove that $f\, g \in L^1(\mathbb{R})$.

Hint: For ii) it might be useful to consider the relation

$$\int_{\mathbb{R}} |f\, g|\, dx = \int_{-1}^{1} |f\, g|\, dx \,+ \sum_{k\geq 0} \int_{2^k\leq|x|\leq 2^{k+1}} |f\, g|\, dx\,.$$

<u>Carefully justify all your reasonings.</u>

Solution Ex.4 (continuation)

ex.5 (25p) Let $\{f_n\}_n$ be a sequence of real-valued Lebesgue measurable functions on $\mathbb{R}$ and let f be another such function. Assume that

(1) $f_n \overset{n\to\infty}{\longrightarrow} f$ Lebesgue almost everywhere;

(2) $\int_{\mathbb{R}} |x\, f_n(x)|\, dx \leq 1 \qquad \forall\, n \in \mathbb{N}$;

(3) $\int_{\mathbb{R}} |f_n(x)|^2\, dx \leq 1 \qquad \forall\, n \in \mathbb{N}$.

Prove that $\{f_n\}_n \subset L^1(\mathbb{R})$, that $f \in L^1(\mathbb{R})$ and that

$$\|f_n - f\|_{L^1(\mathbb{R})} \overset{n\to\infty}{\longrightarrow} 0\,.$$

Moreover, show that neither assumption (2) nor (3) can be omitted in order for the conclusion of our problem to hold.

<u>Justify in detail all the steps in your proof.</u>

Solution Ex.5 (continuation)