MA544 – Qualifying Exam – January 2021

Do not flip this page until you are told to do so

Student id number (do not write your name): ____________________

Instructions:

1. This booklet has 13 pages, and some of them are intentionally left blank.
2. The exam has six problems; the value of each question is stated next to it.
3. Fully and clearly justify your answers. Answers without justifications will not be accepted.
4. Solve one problem per page in the space provided. You may insert pages if necessary, and if you do, clearly state which problem they correspond to. Make sure additional pages are stapled to the booklet.
5. You may not consult any books (physical or online) or notes.
6. You may not handle electronic devices during the exam.
7. You may not discuss the exam with anyone else.
8. The proctor will not answer questions about the exam.

1) Let $(X, \mathcal{M}, \mu)$ be a measure space and let $f : X \longmapsto \mathbb{R}$ be a measurable function. Let

$$\lambda_f(r) = \mu(\{x : |f(x)| > r\}).$$

(you may assume that X is σ-finite if you need.)

a)(10 points) Show that $f \in L^1(X)$ if and only if $\lambda_f \in L^1([0,\infty))$ (with the Lebesgue measure).

b)(10 points) We say that $f \in wL^1(X)$ if $\sup_{r>0} r\lambda_f(r) < \infty$. Show that $L^1(X) \subset wL^1(X)$.

c)(10 points) Show that $wL^1(\mathbb{R}^n) \not\subset L^1(\mathbb{R}^n)$ (with the Lebesgue measure).

Use this page to write your solution to question 1 only.

2) Let $(X, \mathcal{M}, \mu)$ be a measure space.

a) (10 points) True of false (prove of give a counter-example) If $f_n \in L^p(X)$, and $f_n \to f$ in $L^p(X)$, $p \in [1, \infty)$ then $f_n \to f$ in measure.

b)(10 points) True of false (prove of give a counter-example) If $f_n \to f$ a.e. then $f_n \to f$ in measure or locally in measure.

c)(10 points) True of false (prove of give a counter-example) If $f_n \to f$ in measure or locally in measure then $f_n \to f$ a.e.

Use this page to write your solution to question 2 only.

3) a) (10 points) Determine if the following limit exists, and compute it if it does.

$$\lim_{n\to\infty} \int_0^{1-\frac{1}{n}} \left(\sum_{j=0}^{n} (-1)^j \, \frac{x^{j+1}}{j+1} \right) (1+x)^{-2} \, dx$$

b) (10 points) Let

$$F(x) = \int_0^{\infty} \frac{\sin(xy^3)}{(1+y^4)^2} \, dy.$$

Prove that $F(x)$ is differentiable for every x and compute $F'(x)$.

Use this page to write your solution to question 3 only.

4) (30 points) Let $f \in AC([0,1])$ be a non-decreasing function and let $E \subset [0,1]$ be a Lebesgue measurable set. Show that

$$m^*(f(E)) \leq \int_E f'(t)\ dt,$$

where m^* is the Lebesgue outer measure.

Use this page to write your solution to question 4 only.

5) (30 points) We say that $Q \subset \mathbb{R}^n$ is a closed cube if

$$Q = [a_1, b_1] \times [a_2, b_2] \times \ldots [a_n, b_n] \text{ and } b_1 - a_1 = b_2 - a_2 = \ldots = b_n - a_n.$$

The volume of Q is denoted by $V(Q)$. If $B \subset \mathbb{R}^n$ is a ball of radius r, we denote the volume of B by $V(B)$. If $E \subset \mathbb{R}^n$, let

$$m^*(E) = \inf\{\sum_{j=1}^{\infty} V(Q_j) : Q_j \text{ is a closed cube and } E \subset \bigcup_{j=1}^{\infty} Q_j\} \text{ and}$$

$$\mu^*(E) = \inf\{\sum_{j=1}^{\infty} V(B_j) : B_j \text{ is a closed ball and } E \subset \bigcup_{j=1}^{\infty} B_j\}.$$

Prove that $\mu^*(E) \leq m^*(E)$.

(it is true that $\mu^*(E) = m^*(E)$, and you get 10 bonus points if you prove it).

Use this page to write your solution to question 5 only.

6) (30 points) Let $(X, \mathcal{M}, \mu)$ be a measure space and let $f_n \in L^p(X)$, $n \in \mathbb{N}$ and $p \in [1, \infty)$. Suppose that $\lim_{n\to\infty} f_n = f$ a.e. and suppose that for each $\varepsilon > 0$

i) there exists a set A_ε such that $\mu(A_\varepsilon) < \infty$ and $\int_{X \setminus A_\varepsilon} |f_n|^p \, d\mu < \varepsilon$ for all n and that

ii) there exists $\delta > 0$ independent of n such that

$$\int_E |f_n|^p \, d\mu < \varepsilon \text{ provided } \mu(E) < \delta.$$

Prove that $f \in L^p$ and $\lim_{n\to\infty} ||f_n - f||_p = 0$.

Use this page to write your solution to question 6 only.