Real Analysis Qualifying Exam, August 2021.

Instructions: Write your answers to each question in clear, concise, correct English. Each proof you write must contain enough details and should be presented clearly so that the grader can easily follow your argument. Use the space provided for each problem and write on one side of each sheet only. Notes, books, and electronic devices are not allowed. If you need more paper, ask the proctor.

In the problems below, the measure on a subset of $\mathbb{R}$ is Lebesgue measure, which is denoted by m.

Each problem is worth 5 points.

1. Suppose K is a compact metric space and $f_n : K \to \mathbb{R}$ is a sequence of continuous functions converging pointwise to a continuous function f on K. Suppose also that for every x in K and every sequence $\{x_n\}$ converging to x, we have $\lim_{n\to\infty} f_n(x_n) = f(x)$. Prove that f_n converges uniformly to f on K.

(extra page for work)

2. Suppose that f is absolutely continuous on $[0, 1]$ and that $f(x) \neq 0$ for all $x \in [0, 1]$. Prove that $1/f$ is absolutely continuous on $[0, 1]$.

(extra page for work)

3. Suppose that $f \in L^2(\mathbb{R})$, $g \in L^3(\mathbb{R})$, and $h \in L^6(\mathbb{R})$. Prove that fgh is in $L^1(\mathbb{R})$.

(extra page for work)

4. Suppose $f \in L^1([0,1])$ and that $f \geq 0$ a.e. Prove that

$$\lim_{n\to\infty} \int_0^1 f(x)^{1/n}\, dm(x) = m(\{x : f(x) > 0\}).$$

(extra page for work)

5. Let f be Lebesgue measurable on $[0,1]$ with $f(x) > 0$ a.e. Suppose $\{E_k\}$ is a sequence of measurable subsets of $[0,1]$ and that $\int_{E_k} f(x)\,dm(x) \to 0$ as $k \to \infty$. Prove that $m(E_k) \to 0$ as $k \to \infty$.

(extra page for work)

6. Let $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ be measure spaces with both $\mu(X)$ and $\nu(Y)$ positive and finite. Let $f \in L^\infty(\mu)$ and $g \in L^\infty(\nu)$, and suppose that $f(x) = g(y)$ almost everywhere with respect to the product measure $\mu \otimes \nu$. Prove that there is a constant $a \in \mathbb{R}$ such that $f(x) = a$ μ-a.e. and $g(y) = a$ ν-a.e.

(extra page for work)

7. Suppose that $f \in L^p(\mathbb{R})$ for all $p \in (1, 2)$ and that $\sup_{1<p<2} \|f\|_p < \infty$. Prove that $f \in L^2(\mathbb{R})$ and that

$$\lim_{p \uparrow 2} \|f\|_p^p = \|f\|_2^2.$$

(extra page for work)