MA 54400 QUALIFIER EXAM, JANUARY 2020

Each problem is worth 5 points. Make sure that you justify your answers. No books, notes, or any electronic device, please.

1. Suppose $S \subset \mathbb{R}^n$ is bounded and $\operatorname{int} S \neq \emptyset$. Show that there is an $r \in (0, \infty)$ such that S contains some ball $B_r(x)$ of radius r, but it contains no ball $B_R(y)$ of radius strictly greater than r.

2. Let us say that a metric space M has the Lindelöf property if for any collection $G_i \subset M$ $(i \in I)$ of open subsets there is a countable $J \subset I$ such that $\bigcup_{i \in J} G_i = \bigcup_{i \in I} G_i$. Prove that a metric space that has the Lindelöf property must be separable.

3. Suppose $Y \subset \mathbb{R}^n$ has measure zero, and $\phi \colon \mathbb{R}^n \to \mathbb{R}$ is continuous at all points of $\mathbb{R}^n \setminus Y$. Prove that ϕ is measurable.

4. Compute

$$\lim_{k \to \infty} \int_0^1 \frac{k \sin(x/k)}{x^{3/2}}\, dx.$$

5. Consider nonnegative integrable functions h, h_n $(n \in \mathbb{N})$ on a measure space $(\Omega, \mathcal{A}, \mu)$. Assuming

$$\lim_{n \to \infty} h_n = h \quad \text{almost everywhere} \quad \text{and} \qquad \lim_{n \to \infty} \int_\Omega h_n\, d\mu = \int_\Omega h\, d\mu,$$

show that $\lim_{n \to \infty} \int_E h_n\, d\mu = \int_E h\, d\mu$ for all $E \in \mathcal{A}$.

6. Let $q : \mathbb{R} \to \mathbb{R}$ be bounded and measurable, and $Q(x) = \int_0^x q(t)\, dt$ for $x \in \mathbb{R}$ (Lebesgue integral). Show that for any $a < b$

$$\lim_{h \to 0} \int_a^b \left| \frac{Q(x+h) - Q(x)}{h} - q(x) \right| dx = 0 \qquad \text{(Lebesgue integral here, too).}$$

7. In a measure space $(\Omega, \mathcal{A}, \mu)$ consider a bijection $r : \Omega \to \Omega$ such that for all $E \in \mathcal{A}$ we have $r^{-1}(E) \in \mathcal{A}$ and $\mu\big(r^{-1}(E)\big) = \mu(E)$. If $f : \Omega \to [-\infty, \infty]$ is integrable and $f \circ r = -f$, prove that $\int_\Omega f\, d\mu = 0$.

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