REAL ANALYSIS QUALIFYING EXAM
AUGUST 2018

Observation. You have 2 hours to complete this exam. Books, notebooks, and any other course materials are NOT allowed. Cell phones must be turned off. No computers or calculators are accepted. Each problem should be solved on a distinct (new) page (if you need more space ask for supplementary paper). Present your solutions in great detail.

Problem	Points
1	
2	
3	
4	
Total	

1. (10 points) m is the Lebesgue measure on $\mathbb{R}^d$ and Δ denotes symmetric difference, i.e. $A\Delta B = (A \setminus B) \cup (B \setminus A)$.

i) Suppose $f : \mathbb{R}^d \to \mathbb{R}$ is measurable and $\{E_j\}$ is a sequence of measurable subsets of the unit ball such that

$$\chi_{E_j} \to f \qquad \text{a.e.}$$

as $j \to \infty$. Prove that there exists a measurable set $E \subset \mathbb{R}^d$ such that

$$m(E\Delta E_j) \to 0$$

as $j \to \infty$.

ii) If $E \subset \mathbb{R}^d$ is a measurable set such that $m(E) \in [0, \infty)$, show that

$$m(E\Delta(E + t)) \to 0$$

as $|t| \to 0$. Does this hold if $m(E) = \infty$? (justify)

1. continued...

2. (10 points) Suppose $f : [0,1] \to \mathbb{R}$ is absolutely continuous with

$$\int_0^1 e^{|f'|} dx \leq C$$

for some $C > 0$.

i) Prove that

$$\sup_{x \neq y \in [0,1]} \frac{|f(x) - f(y)|}{|x - y|^\alpha} < \infty$$

for all $\alpha \in (0,1)$.

ii) Given $\alpha \in (0,1)$ and $y \in (0,1)$, determine

$$\lim_{x \to y} \frac{|f(x) - f(y)|}{|x - y|^\alpha}.$$

2. continued...

3. (10 points) Let $f \in C(\mathbb{R})$ be a continuous compactly supported function on $\mathbb{R}$ and let

$$P_y(x) = \frac{1}{\pi} \frac{y}{x^2 + y^2} \quad \text{where } x \in \mathbb{R} \text{ and } y > 0\,,$$

be the Poisson kernel for the upper half-plane. Define $u(x, y) := f * P_y(x)$.

Show the following:

i) The Poisson kernel is a good kernel on $\mathbb{R}$ as $y \to 0$.

ii) $u(x, y) \to f(x)$ uniformly as $y \to 0$.

iii) $\int_{\mathbb{R}} |u(x, y) - f(x)|\, dx \to 0$ as $y \to 0$.

You are not allowed to just quote a result. All of the above must be proved in full detail.

3. continued...

4. (10 points) Let $f : \mathbb{R} \to \mathbb{C}$ be a Lebesgue measurable function and set m to be the Lebesgue measure on $\mathbb{R}$. We define the *distribution function* of f as

$$\lambda_f : (0,\infty) \to [0,\infty], \quad \lambda_f(\alpha) := m(\{x : |f(x)| > \alpha\}).$$

Fix $0 < p < \infty$. Show the following:

i)

$$\|f\|_p^p := \int_{\mathbb{R}} |f|^p \, dm = p \int_0^\infty \alpha^{p-1} \lambda_f(\alpha) \, d\alpha .$$

ii) We define the weak-L^p space denoted as $L^{p,\infty}(\mathbb{R})$ as the vector space of all (complex) measurable functions $f : \mathbb{R} \to \mathbb{C}$ such that the quasinorm

$$\|f\|_{p,\infty} := \left(\sup_{\alpha > 0} \alpha^p \lambda_f(\alpha) \right)^{\frac{1}{p}} \quad \text{is finite} .$$

Show that $L^p(\mathbb{R}) \subset L^{p,\infty}(\mathbb{R})$ with $\|f\|_{p,\infty} \le \|f\|_p$.

iii) If $f \in L^{p,\infty}(\mathbb{R})$ and $m(\{x \in \mathbb{R} \,|\, f(x) \neq 0\}) < \infty$ then $f \in L^q(\mathbb{R})$ for any $q < p$. On the other hand, if $f \in L^{p,\infty}(\mathbb{R}) \cap L^\infty(\mathbb{R})$ then $f \in L^q(\mathbb{R})$ for any $q > p$.

4. continued...