Qualifying examination MA544, Fall 2013

Name: ______________________________

1. (15 points) Let $(X, \mathcal{M}, \mu)$ be a measure space and let $f, f_1, f_2, ...$ be non-negative integrable functions such that $f_n \to f$ a.e. and $\lim_{n\to\infty} \int_X f_n d\mu = \int_X f d\mu$. If $E \in \mathcal{M}$, then show that $\lim_{n\to\infty} \int_E f_n d\mu = \int_E f d\mu$.

2. (a) (10 points) Let $f : \mathbb{R} \to \mathbb{R}$ be a Lebesgue integrable function such that $\int_a^b f(x) d\lambda(x) = 0$ for every $a < b$. Show that $f(x) = 0$ for almost every x.

   (b) (5 points) Let $f : \mathbb{R} \to \mathbb{R}$ be a Lebesgue integrable function such that
$$\int_{\mathbb{R}} f(x)g(x)d\lambda(x) = 0$$
   for every continuous function $g : \mathbb{R} \to \mathbb{R}$. Show that $f(x) = 0$ for almost every x.

3. (15 points) Show that $f(x) = \sin(x^2)$ is not Lebesgue integrable over $[0, \infty)$. However, show that the improper Riemann integral $\int_0^\infty \sin(x^2)dx$ exist.

4. (10 points) Let $(X, \mathcal{M}, \mu)$ be a measure space, T a metric space, and $f : X \times T \to \mathbb{R}$ a function. Assume that $f(\cdot, t)$ is a measurable function for each $t \in T$ and $f(x, \cdot)$ is a continuous function for each $x \in X$. Assume also that there exists an integrable function g such that for each $t \in T$ we have $|f(x, t)| \le g(x)$ for almost all $x \in X$. Show that the function $F : T \to \mathbb{R}$, defined by
$$F(t) = \int_X f(x, t)d\mu(x),$$
   is a continuous function.

5. (15 points) Prove the following integral version of Minkowski's inequality for $1 \le p < \infty$: if $f \ge 0$ is a measurable function on $\mathbb{R}^2$, then
$$\left[\int_{\mathbb{R}} \left(\int_{\mathbb{R}} f(x, y)d\lambda(x)\right)^p d\lambda(y)\right]^{1/p} \le \int_{\mathbb{R}} \left(\int_{\mathbb{R}} f(x, y)^p d\lambda(y)\right)^{\frac{1}{p}} d\lambda(x).$$

6. (15 points) Let $(f_n)_{n\ge1}$ be a sequence of absolutely continuous real-valued functions on $[0, 1]$ such that
   (a) $f(x) = \sum_{n=1}^\infty f_n(x)$ converges for every $x \in [0, 1]$.
   (b) $\int_0^1 \left(\sum_{n=1}^\infty |f_n'(x)|\right) d\lambda(x) < \infty$.
   Show that f is absolutely continuous on $[0, 1]$.

7. (15 points) Let $f_n, f \in BV([0, 1]), n \ge 1$. Assume that $\sum_{n=1}^\infty V_{f_n - f}(0; 1) < \infty$. Show that $f_n' \to f'$ λ -a.e. on $[0, 1]$.