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Qualifying exam— Real Analysis

Date: January 2011 *Duration:* 120 min

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PUID: ____

Problem	Points	Score
1	10	
2	15	
3	15	
4	20	
5	20	
6	20	
7	20	
Total:	120	

1. Prove or disprove that $\dfrac{\sin x}{x}$ is Lebesgue integrable on $[0, \infty)$. [10pt]

2. Compute the Lebesgue integral [15pt]

$$\int_0^1 \overline{\lim_{n\to\infty}}\, x\cos(2\pi nx)\,dx.$$

Justify the result.

3. Let $f : \mathbb{R} \to \mathbb{R}$ be a Lebesgue integrable function. Prove that the function [15pt]

$$F(x) = \int_x^\infty f(t)dt$$

is uniformly continuous, starting from the definition of the Lebesgue integral.

4. Show that $F : \mathbb{R} \to \mathbb{R}$ [20pt]

$$F(t) = \int_{-\infty}^{\infty} \frac{\sin(x^2 t)}{1 + x^4}\, dx,$$

is uniformly continuous and diferentiable on $\mathbb{R}$.

5. Let $(a_n)_{n=1}^{\infty}$ be a sequence of real numbers such that $\sum_{n=1}^{\infty} |a_n| < \infty$. [20pt]
Define $f : [-1, 1] \to \mathbb{R}$ by $f(x) = \sum_{n=1}^{\infty} a_n x^n$. Show that f has bounded variation.

Hint: Treat first the case when $a_n \geq 0$.

6. Let $f : [0,1] \to \mathbb{R}$ be absolutely continuous and assume that $f' \in L^2[0,1]$. Show that [20pt]

$$\lim_{x \to 0+} \frac{f(x) - f(0)}{\sqrt{x}} = 0.$$

7. Prove or disprove that $f(x) = \sin^{\alpha} x$ is absolutely absolutely continuous on $[0, 1]$, for any $\alpha \in (0, 1)$. [20pt]