QUALIFYING EXAMINATION
August 2009
MATH 544–Antônio Sá Barreto

Identifier (PLEASE PRINT CLEARLY)

Instructions:

I) This exam booklet contains 6 problems and 13 pages. The value of each question is indicated next to its statement. Write the solutions on this booklet. If you need extra space, paper will be provided. If you use extra paper, please clearly state the question you used it for.

II Answers without justification will not be accepted.

III) No questions are allowed during the exam. If you believe there is something wrong with a particular problem, indicate what it is and/or give a counterexample.

1) Let $f \in L^1(\mathbb{R})$ and let

$$F_\lambda(z) = \sqrt{\frac{\lambda}{\pi}} \int_{\mathbb{R}} e^{-\lambda(z-t)^2} f(t)\, dt, \quad \lambda > 0.$$

a) (10 points) Show that $F_\lambda(\bullet)$ is a $C^\infty(\mathbb{R})$ function of z and in fact it has an extension to an analytic function on $\mathbb{C}$.

b) (30 points) Show that when $z \in \mathbb{R}$,

$$\lim_{\lambda \to \infty} F_\lambda(z) = f(z) \text{ in } L^1(\mathbb{R}).$$

Use that $\int_{\mathbb{R}} e^{-\lambda t^2} dt = \sqrt{\frac{\pi}{\lambda}}$.

Problem #1

2) (30 points) Let $A \subset \mathbb{R}$ be a Lebesgue measurable set. Suppose that there exists $\delta > 0$ such that for each interval $I \subset \mathbb{R}$

$$m(A \cap I) \geq \delta m(I),$$

where m denotes the Lebesgue measure. Show that $m(A^c) = 0$, where $A^c = \mathbb{R} \setminus A$.

Problem # 2

3) Let $A \subset (0,1)$ be a Lebesgue measurable set with $m(A) = 0$. m here indicates the Lebesgue measure.

a) (10 points) Show that for each $k \in \mathbb{N}$ one can pick an open subset $G_k \subset (0,1)$ such that $A \subset G_k$ and that $m(G_k) < 2^{-k}$.

b) (10 points) Let $f_k(x) = m(G_k \cap (0,x))$. Show that f_k is continuous and increasing and that $f_k(x) \leq 2^{-k}$.

c) (20 points) Show that $f(x) = \sum_{k=1}^{\infty} f_k(x)$ is continuous and increasing on $(0,1)$ and that $f'(t)$ is not defined for $t \in A$.

Problem # 3

4) (20 points) Let $U = \{r_j, \quad j \in \mathbb{N}\} \subset [0,1]$ and let $\{a_n, \quad n \in \mathbb{N}\}$ be a sequence of real numbers so that $\sum_{n=1}^{\infty} |a_n| < \infty$. Prove that the series

$$S(x) = \sum_{n=1}^{\infty} \frac{a_n}{|x - r_n|^{\alpha}}, \text{ with } 0 < \alpha < 1 \text{ and } \quad x \notin U,$$

converges absolutely for a.e. $x \in [0,1]$ with respect to the Lebesgue measure.

Suggestion (which you are not obliged to follow): Let $S_N(x) = \sum_{n=1}^{N} \frac{|a_n|}{|x-r_n|^{\alpha}}$, with $x \notin \mathbb{Q}$, and consider $||S_N - S_M||_{L^1([0,1])}$.

Problem # 4

5) (20 Points) Let $f \in L^1(\mathbb{R})$ be continuous. Compute the limit

$$\lim_{n\to\infty} \int_a^\infty \frac{n^{\frac{1}{2}} f(t)}{1+nt^2}\, dt \text{ when } a < 0, a = 0 \text{ and } a > 0.$$

Problem # 5

6) a)(10 points) Show that $f(t) = t^{1/3}$ and $g(t) = t^{3/2}\sin(1/t)$ are absolutely continuous on $[0, 1]$.

b) (10 points) Show that if $a \leq 1$ and $k \in \mathbb{N}$ is odd, the function $g(t) = t^a(\sin(1/t))^{1/k}$ is not of bounded variation on $[0, 1]$.

c)(10 points) Let $f, g : [0, 1] \longrightarrow [0, 1]$ be absolutely continuous functions, show that $f \circ g$ is not necessarily of bounded variation.

Problem # 6