Math 544
Qualifying Examination
January 2008
Prof. N. Garofalo

Name..

I. D. no. ....

Problem	Score	Max. pts.
1		20
2		20
3		25
4		20
5		20
6		25
Total		130

Problem 1. (20 points) Let $f \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ be a function such that for some $s > n/2$ its Fourier transform $\hat{f}(\xi) = \int_{\mathbb{R}^n} e^{-2\pi i <\xi,x>} f(x)dx$ is such that

$$(1) \qquad \int_{\mathbb{R}^n} (1 + |\xi|^2)^s |\hat{f}(\xi)|^2 \, d\xi \ < \ \infty \ .$$

1) Prove that $\hat{f} \in L^1(\mathbb{R}^n)$.
2) You can assume as known that if $f, \hat{f} \in L^1(\mathbb{R}^n)$, then the following inversion formula holds

$$(2) \qquad f(x) \ = \ \int_{\mathbb{R}^n} e^{2\pi i <x,\xi>} \hat{f}(\xi) \, d\xi \ , \qquad \text{for a.e. } x \in \mathbb{R}^n \ .$$

Use (2) and Part 1) to prove that if $f \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ is a function such that (1) holds for some $s > n/2$, then f coincides a.e. with a continuous function on $\mathbb{R}^n$.

Problem 2. (20 points) Prove that if $f(x) = \chi_R(x)$, where $R = (a, b) \times (c, d) \subset \mathbb{R}^2$, and χ_R denotes the characteristic function of R, then the Fourier transform of f verifies $\hat{f}(\xi) \to 0$ as $|\xi| \to \infty$.

Hint: Start by computing the Fourier transform $\hat{\chi}_{(-A,A)}$ of a symmetric interval on the line, then...

Problem 3. (25 points) Consider the function $K(x) = (4\pi)^{-n/2}\exp(-|x|^2/4)$, $x \in \mathbb{R}^n$, and let

$$K_t(x) \;=\; t^{-n/2}K(x/\sqrt{t}) \;, \quad t > 0 \;.$$

Given a measurable function $f : \mathbb{R}^n \to \overline{\mathbb{R}}$, define

$$P_t f(x) \;=\; K_t \star f(x) \;=\; \int_{\mathbb{R}^n} K_t(x-y)f(y)dy \;.$$

1) Prove that for every $1 \le p < \infty$, one has

$$\lim_{t\to 0^+} ||P_t f - f||_{L^p(\mathbb{R}^n)} \;=\; 0 \;, \quad \text{for every } f \in L^p(\mathbb{R}^n) \;,$$

and that moreover for every $1 \le p \le \infty$

$$||P_t f||_{L^p(\mathbb{R}^n)} \;\le\; ||f||_{L^p(\mathbb{R}^n)} \;.$$

2) Prove that if $1 \le p < \infty$ one also has $P_t : L^p(\mathbb{R}^n) \to L^\infty(\mathbb{R}^n)$ with

$$||P_t f||_{L^\infty(\mathbb{R}^n)} \;\le\; (p')^{-n/2p'}(4\pi t)^{-n/2p}||f||_{L^p(\mathbb{R}^n)} \;,$$

where $1/p + 1/p' = 1$. Here, one must take $(p')^{-n/2p'} = 1$ when $p = 1$.

Problem 4. (20 points) Let

$$N(x,y,z) \;=\; ((x^2+y^2)^2+z^2)^{1/4}\ , \quad (x,y,z) \in \mathbb{R}^3\ .$$

Using the change of variable

$$x \;=\; \rho\cos\theta\sin^{1/2}\phi\ , \quad y \;=\; \rho\sin\theta\sin^{1/2}\phi\ , \quad z \;=\; \rho^2\cos\phi\ ,$$

with $(\rho,\theta,\phi) \in \Omega = (0,\infty)\times(0,2\pi)\times(0,\pi)$, identify all values of $p \in \mathbb{R}$ for which $N^{-p} \in L^1(E)$, where $E = \{(x,y,z) \in \mathbb{R}^3 \mid N(x,y,z) < R\}$, with $R > 0$ fixed.

Problem 5. (20 points) Use Lebesgue dominated convergence theorem to compute the limit

$$\lim_{k\to\infty}\int_{B(0,1)}\frac{1-e^{-\frac{|x|^2}{k}}}{|x|^{n+1}}dx\ ,$$

where $B(0,1)=\{x\in\mathbb{R}^n \mid |x|<1\}$.

Problem 6. (25 points) Let $\mathbb{S}^{n-1} = \{\omega \in \mathbb{R}^n \mid |\omega| = 1\}$ be the unit sphere in $\mathbb{R}^n$ and for every $\lambda > 0$ consider the function

$$f(x) \overset{def}{=} \int_{\mathbb{S}^{n-1}} e^{-i\sqrt{\lambda}<x,\omega>} \, d\sigma(\omega) \ , \quad x \in \mathbb{R}^n \ ,$$

where $d\sigma(\omega)$ indicates the induced Lebesgue measure on $\mathbb{S}^{n-1}$.

1) Prove that $f \in C^2(\mathbb{R}^n)$ and satisfies the equation

$$\Delta f \ = \ - \lambda \, f \ ,$$

where $\Delta f = \sum_{j=1}^{n} \frac{\partial^2 f}{\partial x_j^2}$.

2) Prove that if $n = 3$, then

$$f(x) \ = \ 4\pi \frac{\sin(\sqrt{\lambda}|x|)}{\sqrt{\lambda}|x|} \ .$$

Hint: Use spherical coordinates in $\mathbb{R}^3$ and then express the integral on the unit sphere $\mathbb{S}^2 = \{\omega \in \mathbb{R}^3 \mid |\omega| = 1\}$ as an iterated integral over the one-parameter family of circles $L_\theta = \{\omega \in \mathbb{S}^2 \, |< \omega, \frac{x}{|x|} >= \cos\theta\}$ forming an angle θ with the fixed direction $\frac{x}{|x|} \in \mathbb{S}^2$.