Math 544
Qualifying Examination
August 2007
Prof. N. Garofalo

Name..

I. D. no. ....

Problem	Score	Max. pts.
1		20
2		20
3		20
4		20
5		20
6		25
7		25
Total		150

Problem 1. (20 points) **(i)** Let $f \in L^1(0,\infty)$. Prove that there exists a sequence $x_k \nearrow \infty$ such that

$$\lim_{k\to\infty} x_k \, |f(x_k)| \ = \ 0 \ .$$

(ii) Let $f \in L^1(\mathbb{R}^n)$ with $n \geq 2$. Prove that there exists a sequence $R_k \nearrow \infty$ such that

$$\lim_{k\to\infty} R_k \int_{S(0,R_k)} |f| \, d\sigma \ = \ 0 \ ,$$

where $S(0,r) = \{x \in \mathbb{R}^n \mid |x| = r\}$, and $d\sigma$ represents the $(n-1)$-dimensional Lebesgue measure induced on the sphere $S(0,r)$.

Problem 2. (20 points) Prove that if $E \subset \mathbb{R}^n$ is measurable, then for a.e. $x \in \mathbb{R}^n$

$$\lim_{r \to 0} \frac{|E \cap B(x,r)|}{|B(x,r)|} = 1 \,, \quad \lim_{r \to 0} \frac{|(\mathbb{R}^n \setminus E) \cap B(x,r)|}{|B(x,r)|} = 0 \,.$$

Note: Here for a measurable set $A \subset \mathbb{R}^n$ we have indicated by $|A|$ its n-dimensional Lebesgue measure. Also, $B(x,r) = \{y \in \mathbb{R}^n \mid |y - x| < r\}$.

Problem 3. (20 points) **(i)** Prove that a continuous function $T : \mathbb{R}^n \to \mathbb{R}^n$ maps sets F_σ into sets F_σ.
(ii) Prove that a Lipschitz map $T : \mathbb{R}^n \to \mathbb{R}^n$, i.e., a map satisfying $|T(x) - T(y)| \leq L|x - y|$ for every $x, y \in \mathbb{R}^n$, and for some $L > 0$, maps measurable sets into measurable sets. You can take for granted here that a Lipschitz map preserves sets of measure zero.

Problem 4. (20 points) Let $f \in L^p(\mathbb{R}^n), g \in L^{p'}(\mathbb{R}^n)$, with $1 \leq p \leq \infty$ and $1/p + 1/p' = 1$, prove that $f \star g \in C(\mathbb{R}^n)$, where $f \star g(x) = \int_{\mathbb{R}^n} f(x-y)g(y)dy$.

Problem 5. (20 points) Let $f \in L^1_{loc}(\mathbb{R}^n)$, and $0 < \alpha < n$, prove that for every $\epsilon > 0$

$$\left| \int_{B(x,\epsilon)} \frac{f(y)}{|x-y|^{n-\alpha}} dy \right| \ \leq \ C(n,\alpha)\epsilon^{\alpha} \ Mf(x) \ ,$$

where $Mf(x) = \sup\limits_{r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y)| dy$ indicates the Hardy-Littlewood maximal function of f, and $|B(x,r)|$ denotes the n-dimensional Lebesgue measure of the ball $B(x,r) = \{y \in \mathbb{R}^n \mid |y-x| < r\}$.
Hint: Split the region of integration into the union of dyadic rings $\epsilon/2^{k+1} \leq |y-x| < \epsilon/2^k$.

Problem 6. (25 points) For every $\xi \in \mathbb{R}^n$ prove the existence of the limit

$$\lim_{k\to\infty} \int_{\mathbb{R}^n} \frac{e^{-2\pi i<\xi,x>} e^{-|x|^2}}{\left(\frac{1}{k^n} + k|x|^2\right)^{\frac{n+1}{2}}} dx \ ,$$

and compute it explicitly in terms of the $(n-1)$-dimensional measure σ_{n-1} of the unit sphere in $\mathbb{R}^n$, and of the Beta function

$$B(x,y) \stackrel{def}{=} 2\int_0^{\frac{\pi}{2}} (\cos\theta)^{2x-1} (\sin\theta)^{2y-1} d\theta \ , \quad x,y > 0 \ .$$

Remark: Note carefully that you are not required to know or write explicitly the value of σ_{n-1}.

Problem 7. (25 points) Let $f \in L^1(\mathbb{R}^3)$ and assume that $f(x) = f^\star(|x|)$. Prove that for every $\xi \in \mathbb{R}^3$

$$\hat{f}(\xi) \overset{def}{=} \int_{\mathbb{R}^3} e^{-2\pi i <\xi, x>} f(x)\ dx \ = \ \frac{2}{|\xi|} \int_0^\infty f^\star(r) r \sin(2\pi r |\xi|)\ dr\ .$$

Hint: Use spherical coordinates in $\mathbb{R}^3$ and then express the integral on the unit sphere $\mathbb{S}^2 = \{\omega \in \mathbb{R}^3 \mid |\omega| = 1\}$ as an iterated integral over the one-parameter family of circles $L_\theta = \{\omega \in \mathbb{S}^2 \mid <\omega, \frac{\xi}{|\xi|}> = \cos\theta\}$ forming an angle θ with the fixed direction $\frac{\xi}{|\xi|} \in \mathbb{S}^2$.