MA 544 Qualifying Exam, Fall 2006

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Note: All measures are Lebesgue measure. The Lebesgue measure of a set A is $m(A)$.

1. Let $f_n : \mathbb{R} \mapsto (-\infty, \infty]$, $n \in \mathbb{N}$, be a sequence of Lebesgue measurable functions.

   (a) Prove that $g(x) = \sup_n f_n(x)$ is Lebesgue measurable. 5 pts.

   (b) Prove that $h(x) = \limsup_{n\to\infty} f_n(x)$ is Lebesgue measurable. 5 pts.

2. Suppose that f_n, $n \in \mathbb{N}$, is a sequence of integrable functions on $[0, 1]$ such that (a) $\lim_{n\to\infty} f_n(x) = 0$ for all $x \in [0, 1]$ and (b) $\int_0^1 f_n(x)\, dx = 0$ for all n. Does it follow that $\lim_{n\to\infty} \int_0^1 |f_n(x)|\, dx = 0$? Either give a proof or a counter example. 10 pts.

3. Find the following limits and prove your answers.

   10 pts.

   (a)

$$\lim_{t\to 0^+} \int_0^1 \frac{e^{-t \ln x} - 1}{t}\, dx$$

   10 pts.

   (b)

$$\lim_{n\to\infty} \int_1^{n^2} \frac{n \cos\left(\frac{x}{n^2}\right)}{1 + n \ln x}\, dx$$

4. Suppose that $f_n(x)$ is a sequence of increasing (in x), absolutely continuous functions on $[0, 1]$ for which $f_n(0) = 0$ for all n. Let

$$g(x) = \sum_1^\infty f_n(x).$$

   Prove that if $g(1) < \infty$, then g is absolutely continuous on $[0, 1]$. 20 pts.

5. For f a measurable, real valued function on $\mathbb{R}^+$, let

$$T(f)(x) = \int_1^\infty \frac{1}{\sqrt{u} + 1 + x^2} f(u)\, du$$

whenever the function appearing in the integrand is integrable with respect to u. Let $1 < q < 2$ be fixed.

(a) Prove that $T(f)(x)$ is defined for all $x \in \mathbb{R}$ if $f \in L^q(\mathbb{R}^+)$. 5 pts.

(b) Prove that there is a constant C_q, independent of f, x, and y, such that for all x and y 5 pts.

$$|T(f)(x) - T(f)(y)| \leq C_q |x^2 - y^2| \|f\|_q.$$

(c) Let $K \subset \mathbb{R}$ be compact and let $C(K)$ be the set of continuous functions g on K with norm 10 pts.

$$\|g\| = \sup_{x \in K} |g(x)|.$$

Show that the set $S = \{T(f)\big|_K \mid \|f\|_q \leq 1\}$ has compact closure in $C(K)$. ($T(f)\big|_K$ denotes the restriction of $T(f)$ to K.)

6. Suppose that $f_n \in L^1[0,1]$, $n \in \mathbb{N}$, is such that $\lim_{n\to\infty} f_n(x) = f(x)$ a.e. x.

(a) Suppose that $\lim_{n\to\infty} |f_n(x)|^{\frac{1}{p}} = |f(x)|^{\frac{1}{p}}$ *uniformly* on $[0,1]$. Prove that then $\lim_{n\to\infty} |f_n|^{\frac{1}{p}} = |f|^{\frac{1}{p}}$ in $L^p[0,1]$–i.e. prove that $\lim_{n\to\infty} \||f_n|^{\frac{1}{p}} - |f|^{\frac{1}{p}}\|_p = 0$. 5 pts.

(b) Prove that the conclusion in 6a still holds if instead of the uniform convergence of $|f_n|^{\frac{1}{p}}$, we assume that $\lim_{n\to\infty} f_n = f$ in $L^1[0,1]$–i.e. $\lim_{n\to\infty} \|f_n - f\|_1 = 0$. 15 pts.

In your proof you may assume the following "well known" result: If $f \in L^1([0,1])$, then for all $\epsilon > 0$, there is a $\delta > 0$ such that $m(A) < \delta$ implies that $\int_A |f(x)|\, dx < \epsilon$.

Remark. The result in 6b is actually true under the weaker hypothesis that $\lim_{n\to\infty} \int_0^1 |f_n(x)|\, dx = \int_0^1 |f(x)|\, dx$ rather than $\lim_{n\to\infty} f_n = f$ in $L^1[0,1]$. The proof of this more general result is somewhat more complicated and is not requested.