QUALIFYING EXAMINATION
AUGUST 2002

MATH 544 - Prof. Neugebauer

Name: ______________________________

Instructions. Standard notation is used throughout or will be explained in the statement of each problem. There will be 6 *additional* pages with a problem on each page. Use the space provided for your solution of the problem.

1. Let $f \in L^1([0,1])$, and let $F(x) = \int_0^x f(t)dm$. If E is a measurable subset of $[0,1]$, show that
   (i) $F(E) = \{y : \exists x \in E \quad \text{with } y = F(x)\}$ is measurable.

   (ii) $m\{F(E)\} \leq \int_E |f(t)|dm$.

2. Let $f : \mathbb{R}_+ \to \mathbb{R}_+, \mathbb{R}_+ = \{x \in \mathbb{R} : x \geq 0\}$. Assume that $f \in L^1(\mathbb{R}_+)$ satisfies

$$f(x) \leq c \int_0^x f(t)dm(t)$$

with c independent of x. Show that $f(x) = 0$ for every $x \in \mathbb{R}_+$.
(Hint: Write $f(x)$ as iterated integrals.)

3. Let $(X, \mathcal{M}, \mu)$ be a measure space. Two functions $f, g : X \to \mathbb{R}$ are said to be *co-monotone* if

$$[f(x) - f(y)][g(x) - g(y)] \geq 0 \quad \text{for every } (x, y) \in X \times X.$$

If $\mu(X) = 1$ and $f, g \in L^1(\mu)$ are co-monotone, show that

$$\int_X f d\mu \cdot \int_X g d\mu \leq \int_X f g d\mu.$$

4. Let $(X\mathcal{M},\mu)$ be a measure space and let $1 < p < \infty$. Assume that $f : X \to \mathbb{R}$ is $\mathcal{M}$-measurable and satisfies

$$\mu\{x : |f(x)| > y\} \leq \frac{c_0}{y^p}, \quad c_0 \text{ independent of } y > 0.$$

Let $1 \leq r < p$. Show that

$$\int_X |f|^r d\mu \leq c\mu(X)^{1-r/p},$$

where c depends only on c_0, r, p.
(Hint: The left side $= r\int_0^\infty \cdots dy = r\int_0^R + r\int_R^\infty$.)

5. Let $G_1 \supset G_2 \supset \cdots$ be a sequence of open sets in $I = [0,1]$ and let

$$f(x) = \sum_{n=1}^{\infty} m([0,x] \cap G_n).$$

(i) Show that $f \in AC(I)$ if and only if $\sum_{n=1}^{\infty} m(G_n) < \infty$.

(ii) Show that $f \in Lip(I)$, i.e., there exists $0 < M < \infty$ such that $|f(x) - f(y)| \leq M|x-y|$ for all $x, y \in I$, if and only if there exists $N \in \mathbb{N}$ such that $G_n = \emptyset$, $n \geq N$.

6. Let $I = [a, b]$ and let $f : I \to \mathbb{R}$ satisfy for some $1 < p < \infty$

$$V_p(f; I) = \sup_{\Delta} \sum_{j=1}^{n} \frac{|f(x_j) - f(x_{j-1})|^p}{|x_j - x_{j-1}|^{p-1}} < \infty,$$

where the sup is extended over all partitions $\Delta = \{a = x_0 < x_1 < \cdots < x_n = b\}$.
(i) Show that $f \in AC(I)$.

(ii) Show that $f' \in L^p(I)$.
(Hint: $f_n = \sum_{j=1}^{n} \frac{f(x_j)-f(x_{j-1})}{|I_j|} \chi_{I_j}(x)$, where $x_j = a + j|I|/n, I_j = [x_{j-1}, x_j]$, $j = 1, 2, \cdots n$).