QUALIFYING EXAMINATION
JANUARY 2000
MATH 544 - Prof. Lempert

1. Show that a bounded function $f\colon [0,1] \to \mathbb{R}$ is continuous if and only if its graph $\{(x, f(x)) \colon x \in [0,1]\}$ is a closed subset of $\mathbb{R}^2$.

2. Is there a sequence of positive numbers $\varepsilon_1, \varepsilon_2, \ldots$ with the following property? If $g_n \in C[0,1]$ and $g_n \to 0$ pointwise, then $\varepsilon_n g_n \to 0$ uniformly.

3. Suppose f_n are absolutely continuous on $[a,b]$, $n \in \mathbb{N}$, and $f\colon [a,b] \to \mathbb{R}$ is such that $\lim_{n\to\infty} T(f - f_n) = 0$. Here T denotes total variation on $[a,b]$. Prove that f is also absolutely continuous.

4. Show that if $F \subset \mathbb{R}$ is a closed set then ∂F has no interior point.

5. Suppose f_n are measurable functions on some measure space $(\Omega, \mathcal{A}, \mu)$, $n = 1, 2, \ldots$, and $\sum_{n=1}^{\infty} \mu\{x \in \Omega \colon |f_n(x)| > 1/n\} < \infty$. Prove that $f_n \to 0$ a.e.

6. Let $(\Omega, \mathcal{A}, \mu)$ be a measure space, $1 \le p < q \le \infty$, and $h \in L^p(\Omega, \mathcal{A}, \mu) \cap L^q(\Omega, \mathcal{A}, \mu)$. Prove that $h \in L^r(\Omega, \mathcal{A}, \mu)$ for every $r \in [p, q]$.

7. Suppose $E \subset \mathbb{R}$ has finite Lebesgue measure and $\varphi \in L^1(\mathbb{R})$. Show that
$$\lim_{t\to\infty} \int_E \varphi(x+t)\,dx = 0.$$

8. Suppose $1 \le p < \infty$, ψ_n, $\psi \in L^p[0,1]$, $\psi_n \ge 0$ a.e., $\lim_{n\to\infty} \psi_n = \psi$ a.e., and $\lim_{n\to\infty} \int_0^1 \psi_n^p = \int_0^1 \psi^p$. Prove that $\psi_n \to \psi$ in $L^p[0,1]$.