QUALIFYING EXAMINATION
August 1999
MATH 544 - Prof. Hunt

This test consists of eight questions on two pages.

You do not need to formally state definitions and theorems that you use, but their exact statements should be clear from your work. Be sure that you indicate that all conditions of any theorem that you use are satisfied.

Verify any statements that you make.

"Discuss" means to state different possibilities that could occur. State conditions that imply particular results. Give proofs and counterexamples.

Graphs may be used to indicate functions that give counterexamples.

1. Suppose that f_n is real-valued and nondecreasing on $[0,1], n \geq 1$, and that $f_n(r) \to f(r)$ for every rational number r in $[0,1]$. Show that $f_n(c) \to f(c)$ at every point c where f is continuous.

2. Suppose that A_n is a Lebesgue measurable subset of R, $n \geq 1$.

(a) Show that $m\left\{\bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} A_k\right\} \leq \liminf_{n\to\infty} m(A_n)$.

(b) Find a sequence of sets A_n for which equality does not hold in (a).

(c) Discuss the inequality $m\left\{\bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k\right\} \geq \limsup_{n\to\infty} m(A_n)$.

3. Suppose that f is Lebesgue measurable and $0 \leq f(x) \leq 1$ on $[0,1]$, and let $\lambda(y) = m\{x \in [0,1] : f(x) > y\}, y \geq 0$, where m denotes Lebesgue measure.

(a) Show that λ is nonincreasing and continuous from the right on $[0,\infty)$.

(b) Show that approximations of the Lebesgue integral $\int_0^1 f\,dm$ are Riemann sums of $\int_0^1 \lambda(y)\,dy$.

4. Show that $f \in L^1(R)$ implies $\lim_{y\to\infty} y\, m\{|f| > y\} = 0$.

5. Suppose that f is real-valued and nondecreasing on $[0,1]$, and

$$\limsup_{y\to x^+} \frac{f(y)-f(x)}{y-x} < u \text{ for } a.e.\ x \in [0,1],$$

where u is a fixed positive number. Also, for a fixed positive number v, suppose that there exists a $\delta > 0$ such that $\sum_{n=1}^{N} (f(b_n) - f(a_n)) < v$, whenever $\{[a_n, b_n]\}_{n=1}^{N}$ is any finite collection of disjoint subintervals of $[0,1]$ with $\sum_{n=1}^{N} (b_n - a_n) < \delta$. Show that $f(1) - f(0) < u(b-a) + v$.

6. Suppose that $\sum_{n=1}^{\infty} \|f_n\|_p < \infty$ for some $1 \le p < \infty$. Show that the series $\sum_{n=1}^{\infty} f_n$ converges $a.e.$ and in L^p norm.

7. Discuss the convergence of the sequence $\left\{\int_0^1 x^{-1/n}|f(x)|\,dx\right\}_{n=1}^{\infty}$ for $f \in L^1[0,1]$.

8. Evaluate $\lim_{n\to\infty} \int_0^{\infty} \frac{n}{e^x + n^2 x}\,dx$. Justify your answer.