QUALIFYING EXAMINATION
MA 544

FALL 1995

Name: ______________________________

Instructions. Standard notation is used throughout. In particular, $\mathbb{R}$ ={reals}, $\mathbb{R}_+ = \{x \in \mathbb{R} : x \geq 0\}$, $I_0 = [0,1]$, and $C(I_0), BV(I_0), AC(I_0), L^p(I_0)$ are the common function spaces over I_0. For a measurable subset A of $\mathbb{R}$, let $|A|$ denote the Lebesgue measure of A. All functions are assumed to be measurable.

There will be 6 *additional* pages with a problem on each page. Use the space provided for your solution of the problem.

1. Let $f, g : \mathbb{R}_+ \to \mathbb{R}_+$ have the property that for each $0 < x < \infty$

$$\int_0^x g(t)\, dt \leq \int_0^x f(t)\, dt.$$

Show that, if $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ is non-increasing, then

$$\int_0^\infty \phi(t) g(t)\, dt \leq \int_0^\infty \phi(t) f(t)\, dt.$$

2. Let $f \in L^p(I_0)$ for some $1 \leq p < \infty$. Assume that for all $x \in I_0$

$$\int_{I_0} f(y) \sin(xy) \, dy = 0.$$

Show that $f(x) = 0$ for a.e. $x \in I_0$.

3. Let $\{f_n\} \subset L^p(I_0)$ for some $1 \leq p < \infty$, and assume that $f_n \to f(L^p)$.

(i) Give an example showing that there may not exist $g \in L^p(I_0)$ such that $|f_n(x)| \leq g(x)$ for a.e. $x \in I_0$ and $n = 1, 2, \cdots$.

(ii) Show that there exists $n_1 < n_2 < \cdots$ and there exists $g \in L^p(I_0)$ such that $|f_{n_j}(x)| \leq g(x)$ for a.e. $x \in I_0$ and $j = 1, 2, \cdots$.

4. Let f, g be two non-negative functions belonging to $L^p(I_0)$ for some $1 < p < \infty$. Assume that for every $0 < y < \infty$,

$$|\{x : g(x) > y\}| \leq \frac{1}{y} \int_{\{x:g(x)>y\}} f(x)\, dx.$$

Show that $||g||_p \leq p'||f||_p$, $1/p + 1/p' = 1$.

Hint:(i) you may use $||\phi||_p^p = p \int_0^\infty y^{p-1}|\{x : |\phi(x)| > y\}|dy$, and (ii) iterated integrals are equal.

5. Let $f, g : I_0 \to I_0$ be in $AC(I_0)$.
   (i) Give an example showing that $f \circ g$ need not be in $AC(I_0)$.
   (ii) Prove: If $f \circ g \in BV(I_0)$, then $f \circ g \in AC(I_0)$.

6. Let $f \in C(I_0) \cap BV(I_0)$. Show that there exists a homeomorphism h from I_0 onto I_0 such that $f \circ h$ is Lipschitz on I_0, i.e., there exists $0 < M < \infty$ such that for every $x', x" \in I_0$ we have $|f \circ h(x') - f \circ h(x")| \leq M|x' - x"|$.

Hint: Let $v(t) =$ variation of f on $[0, t]$, and let h be the inverse of $\phi(t) = c(t + v(t))$ for some suitable constant c.