MA 530 Qualifying Exam
August 2023

Each problem is worth 10 points. Justify your answers.

1. Let f be an unbounded entire function and $\Omega \subset \mathbb{C}$ a nonempty open set. Show that there exists $p \in \mathbb{C}$ such that $f(p) \in \Omega$.

2. Let $u\colon \mathbb{C} \to \mathbb{R}$ be a harmonic function. Prove that u is either surjective or constant.

3. Find all entire functions f such that $|f(z)| \leq |z|$ for all z and $f(i) = 1$.

4. Evaluate
$$\int_\gamma f(z)dz,$$
where $f(z) = \tan((1+i)z)$, and γ is the circle $|z| = 2$, oriented clockwise.

5. Let $\Omega = \mathbb{C} \setminus \{z \in \mathbb{R}\colon z \leq 0\}$. Find a bijective holomorphic function $f\colon \Omega \to \Omega$ such that $f(1) = i$.

6. Let $f(z) = z^{1000} + z^{100} + z^{10} + 1$. Find an $R > 0$ such that if $f(z) = 0$ then $R < |z| < R+1$.

7. Let $\Omega \subset \mathbb{C}$ be a nonempty open set, and let $f\colon \Omega \to \mathbb{C}$ be holomorphic. Suppose that for every $z \in \Omega$ there is a positive integer n such that $f^{(n!)}(z) = 0$. Prove that f is a polynomial.