MA 530 Qualifier Exam, January 3, 2023

Each of the seven problems below is worth 5 points. In the problems D stands for the unit disc $D = \{z \in \mathbb{C} : |z| < 1\}$.

In your solutions make sure you justify your claims. Notes, books, cribsheets, and electronic devices are not allowed. Efforts to write neatly will be appreciated. The order of the problems is alphabetical, and is not intended to indicate their levels of difficulty.

1. Compute

$$\int_{|z|=2} \frac{e^{iz}\, dz}{4z^2 - \pi^2},$$

where the path of integration is oriented counterclockwise.

2. For a natural number n, let T_n denote the polynomial

$$T_n(z) = 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - + \cdots + (-1)^n \frac{z^{2n}}{(2n+1)!}.$$

Prove that there is an n_0 such that T_n has exactly 6 roots in the disc $\{z \in \mathbb{C} : |z| < 10\}$ when $n > n_0$.

3. If $\cos z = \cos w$ for some complex numbers z, w, prove that there is an integer k such that $z = w + 2k\pi i$ or $z = -w + 2k\pi i$.

4. Is there a harmonic function $u : D \to \mathbb{R}$ such that $\lim_{z \to \zeta} u(z) = \infty$ for every $\zeta \in \partial D$?

5. Let $Q = \{z \in \mathbb{C} : \mathrm{Re} z, \mathrm{Im} z > 0\}$ stand for the first quadrant. Find a biholomorphic map $F : Q \to Q$ such that $F(2+i) = 1+2i$. (Recall that a biholomorphic map is a surjective holomorphic map with a holomorphic inverse.)

6. Suppose g is a holomorphic function on some domain, and $1/\bar{g}$ is also holomorphic there. Prove that g is constant.

7. Suppose $U \subset \mathbb{C}$ is an open set and $h_j : U \to D$ are holomorphic functions, $j \in \mathbb{N}$, that converge as $j \to \infty$ at each point of a certain subset $A \subset U$. Prove that there is a holomorphic function $h : U \to \mathbb{C}$ such that $\lim_{j \to \infty} h_j = h$ on A.