MATH 530 Qualifying Exam

August 2020 (S. Bell and L. Lempert)

Each problem is worth 20 points

1. Compute
$$\int_{-\infty}^{\infty} \frac{1}{(x^2+1)(x^2+4)}\, dx.$$

2. Suppose $f(z)$ is analytic on $\Omega = \{z : |z| > 2\}$ and maps Ω into itself.

a) Could such an f have an essential singularity at infinity? Explain

b) If $\lim_{z\to\infty} f(z) = \infty$ (i.e., if f has a pole at infinity), prove that $|f(z)| \geq |z|$ for all z in Ω.

3. Find all points in the closed unit disc $\{z : |z| \leq 1\}$ where $|z^3 + z^9|$ assumes its maximum value.

4. Suppose f is analytic on $\mathbb{C} - \mathcal{A}$ where $\mathcal{A}$ is a finite set and that the sum of the residues of f at the points in $\mathcal{A}$ is zero. Prove that f has an analytic antiderivative on $\{z : |z| > R\}$ where $R = \max\{|a| : a \in \mathcal{A}\}$.

5. Let $\Omega = \mathbb{C} - \sigma$ where σ is the set $\{t + i\sin t : t \geq 0\}$. Show that there is a bounded analytic function on Ω that is *not* constant.

6. The Poisson kernel for the unit disc is
$$P(z,\theta) = \frac{1}{2\pi}\,\frac{1-|z|^2}{|e^{i\theta}-z|^2} = \frac{1}{2\pi}\,\mathrm{Re}\left[\frac{e^{i\theta}+z}{e^{i\theta}-z}\right].$$

If $\phi(\theta)$ is a real valued continuous function on $[0, 2\pi]$ with $\phi(0) = \phi(2\pi)$ that vanishes on a nonempty subinterval (α, β), show that the solution $u(z)$ to the Dirichlet problem on the unit disc with boundary data ϕ extends to be harmonic on $\mathbb{C} - \sigma$ where σ is the circular arc parametrized via $\{e^{i\theta} : \beta \leq \theta \leq 2\pi + \alpha\}$ (i.e., the points on the unit circle *outside* the arc with angle between α and β.) Show that if $u(z)$ extends to be harmonic on all of $\mathbb{C}$, then ϕ must be identically zero.