MATH 530 Qualifying Exam

January 2015 (S. Bell)

Each problem is worth 20 points

1. Explain how one can obtain the identity

$$\sum_{n=1}^{\infty} \frac{z^n}{n} = -\text{Log}\,(1-z) \qquad (*)$$

for z near zero from the well known identity,

$$\sum_{n=0}^{\infty} z^n = \frac{1}{1-z}.$$

(Here, Log denotes the Principal branch of the complex logarithm.) Use $(*)$ to obtain a closed expression for $\sum_{n=1}^{\infty} \frac{r^n \sin(n\theta)}{n}$ where $0 \le r < 1$. Explain at least one thing you would have to worry about if you were to try to let $r \to 1$ in this expression. (This is a place where Complex analysis and Fourier analysis meet.)

2. a) State Rouché's Theorem for the unit disc.

b) Prove that if $P(z) = z^n + \sum_{k=0}^{n-1} a_k z^k$ is a complex monic polynomial of degree $n \ge 1$ such that $|a_0| < 1$, then $P(z)$ has at least one zero inside the unit disc. (Rouché's Theorem does not apply here.)

3. Compute $\displaystyle\int_0^{\infty} \frac{x^{\alpha}}{x^3+1}\, dx$ where $0 < \alpha < 1$.

4. Suppose that a sequence of real valued harmonic functions u_n on a domain Ω converges uniformly on compact subsets to a function u.

a) Prove that u is harmonic on Ω.

b) Outline a proof that the first partial derivatives of the u_n converge uniformly on compact subsets to the corresponding derivatives of u. (Details involving calculations can be omitted.)

c) Suppose that Ω is simply connected and that v_n are harmonic conjugates for the u_n. Prove that if $v_n(a)$ converges for some point $a \in \Omega$, then the v_n converge uniformly on compact subsets of Ω to a harmonic conjugate for u.

5. Think of your paper as the complex plane. Draw a cycle Γ that includes two components: a circle oriented counterclockwise and a figure eight that is contained well inside the circle, oriented so that the top loop is counterclockwise and the bottom loop is clockwise. Draw arrows on your curves to show the orientation. Indicate the values of $\text{Ind}_{\Gamma}(z)$ in each region of the plane determined by Γ. If f is analytic on the complex plane minus a line segment L that does not intersect Γ, when can you say with certainty that $\int_{\Gamma} f\, dz = 0$? Why?