MATH 530 Qualifying Exam

August 2011 (S. Bell)

Each problem is worth 20 points

1. Compute $\displaystyle\int_{-\infty}^{\infty} \frac{e^{ix}}{(1+x^2)^2}\, dx.$

2. Find a one-to-one conformal map from the eighth disc

$$\{z = re^{i\theta} : 0 < r < 1, 0 < \theta < \pi/4\}$$

onto the strip $\{z : 0 < \text{Im } z < 1\}$.
(You may express your answer as a composition of more elementary maps.)

3. Let $\log z$ denote the Principal Branch of the complex log function.
a) Show that the Taylor series for $\log\left(\frac{1}{1-z}\right)$ is

$$\sum_{n=1}^{\infty} \frac{z^n}{n}.$$

b) Show that the Taylor series for $\left[\log\left(\frac{1}{1-z}\right)\right]^2$ is

$$\sum_{n=2}^{\infty} \frac{2}{n}\left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n-1}\right) z^n.$$

4. a) State Liouville's Theorem.
b) Prove Liouville's Theorem using only the fact that

$$f'(a) = \frac{1}{2\pi i}\int_{C_R} \frac{f(z)}{(z-a)^2}\, dz,$$

where C_R is the counterclockwise boundary curve of a circle of radius R, a is a point inside the circle, and where f is analytic on a bigger disc containing the circle.
c) Characterize the entire functions f that satisfy the estimate

$$|f(z)| \le |z|^2$$

for all z. Explain.

5. Prove that $z^2 + z$ has an analytic square root on $\{z : |z| > 1\}$.